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Theory of Matrices over any Division Algebra

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I. Introduction.

This paper is concerned with square matrices of order n having for elements numbers of any associative division algebra D, *viz.*, an algebra in which every number d, not zero, has an inverse d^{-1}.

On defining the left linear dependence of the rows and the right linear dependence of the columns, of this matrix with respect to D we are led to the consideration of the row- and the column-ranks of the matrix which we later show to be the same. This is most easily done by reducing it to a diagonal matrix, *i.e.*, one having certain initial diagonal elements equal to 1 and all the remaining elements zero.

Owing to the lack of commutativity of multiplication as regards these elements of the algebra D a non-singular matrix μ will be defined in two ways and the interdependence of these two methods shown. That this non-singular matrix, with elements in D, has an inverse together with its converse, will be shown. For the purposes of reduction we need five elementary transformations which will be defined in *Article 3*. At the same time we must show that the row- and the column-ranks will be unaltered by these elementary transformations.

It will also be shown that the rank of a product matrix cannot exceed the rank of either of its factors, and also that the rank of a matrix is unaltered when multiplied on the left and on the right by non-singular matrices with elements in the algebra under discussion.

In *Article 5* row-and column-equivalence, in D, of two matrices will be defined. This is a step toward defining a matrix, with elements in D, to be equivalent to another with like elements if it is possible to

pass from one to the other by a finite number of elementary transformations. That two matrices, equivalent in D, will have equal rank, as well as the converse of this will be shown in the closing of the discussion of the Theory of Matrices over any Division Algebra.

II. *Linear Dependence.*

There is said to be a left linear dependence, with respect to D, between the rows of the matrix μ, with elements in the algebra D, if there exist elements c_i (not all zero) of D such that

$$(1) \qquad \sum_{i=1}^{n} c_i\, d_{ij} = 0 \qquad (j=1\ldots\ldots n)$$

We may also say that there is a right linear dependence with respect to D between the columns of the matrix μ, with elements in D, if there exists elements b_j (not all zero) of D such that

$$(2) \qquad \sum_{j=1}^{n} d_{ij}\, b_j = 0 \qquad (i=1\ldots\ldots n).$$

If no such numbers c_i and b_j of the algebra D exist then the rows are left linearly independent with respect to D as to rows, and the columns are right linearly independent with respect to D as to columns

Defination :—If a row of a matrix μ, with elements in D, is lef linearly dependent with respect to D on some other row or rows of μ then μ is called singular.

If a column of a matrix μ, with elements in D, is right linearly dependent with respect to D on some other column or columns of μ, then μ is called singular.

If the rows of a matrix μ, with elements in D, are left linearly independent with respect to D, and the columns are right linearly independent with respect to D, then μ is called non-singular.

If τ of the rows of this matrix are left linearly independent with respect to D but if the $(\tau+1)$st......nth rows of them are left linearly dependent on these τ rows with respect to D, then we say that the matrix μ has row rank τ, which we could henceforth denote by r_r.

If σ of the columns of the matrix μ are right linearly independent with respect to D but if the $(\sigma+1)$st.........nth columns of them are linearly dependent on those σ columns with respect to D then we say that the matrix μ has column rank σ which we could henceforth denote by r_c.

III. *Elementary Transformations of matrices over any Division Algebra.*

By certain elementary transformations on the rows or on the columns of a matrix we are able to reduce a matrix with elements in an algebra D, to a diagonal matrix, *i.e.*, one having certain initial diagonal elements equal to 1 and all the remaining elements zero.

Owing to the non-commutativity of multiplication of the elements in this general algebra we use the five elementary transformations as defined in *Algebras and their Arthmetics.**

(i) The addition to the elements of the ith row of the products of any element k, of the algebra D, into the corresponding elements of the jth row, k being used as a left factor, *i.e.*, we use as a left factor a matrix of the same order n, as μ, differing from the identity matrix by having the zero of the ith row and the jth column replaced by $k (i \neq j)$. Call this matrix $a_k^{(ij)}$. Similarly, for subtraction we replace the zero of the ith row and the jth column by $-k$ and get a matrix that we shall call $a_{-k}^{(ij)}$.

At this point, on account of the important role that the inverse matrix plays in this section, we will define it to be that matrix of order n which multiplied by another matrix of the same order either as a right or as a left factor will give as a result the identity matrix of order n.

It is to be noted that $a_k^{(ij)} a_{-k}^{(ij)} = \mathrm{I} = a_{-k}^{(ij)} a_k^{(ij)}$. Hence $a_k^{(ij)}$ and $a_{-k}^{(ij)}$ are inverses, one of the other.

(ii) The addition to the elements of the ith column of the product of the corresponding elements of the jth column into any element k, of the algebra D, k being used as a right factor, *i.e.*, we use the matrix $b_k^{(ij)}$ as a right factor in the case of addition.

This matrix differs from the identity matrix by having the zero of the ith row and the jth column $(i \neq j)$ replaced by k. The matrix $b_{-k}^{(ij)}$ is used for subtraction. Likewise, as the first transformation, $b_k^{(ij)} \times b_{-k}^{(ij)} = \mathrm{I} = b_{-k}^{(ij)} b_k^{(ij)}$. Hence $b_k^{(ij)}$ and $b_{-k}^{(ij)}$ are inverses, one of the other.

* By Prof. L. E. Dickson Chicago (1923).

(*iii*) *The interchange of any two rows or of any two columns.*

We select a matrix c of order n with 1's in the secondary diagonal and zeros elsewhere. If c is used as a left factor in its product with μ, the first and the nth rows of μ will be found interchanged in μc, the second and the $(n-1)$st rows of μ will be found interchanged in $c\mu$, etc. Should n be odd the middle row remains fixed. If this c is used as a right factor in its product with μ then the first and the nth columns of μ will be found interchanged in μc, etc.

Using the identity matrix and interchanging the ith and the jth rows of it before using it as a left factor we will find the ith and the jth rows of $c_1\ \mu\ (i \neq j)$ interchanged. The same holds as to columns when c is used as a right factor.

$c^2 = \mathrm{I}$, making c its own inverse. Likewise, $c_1^2 = \mathrm{I}$, making c_1 its own inverse.

(*iv*) *The insertion of the same factor* λ, *from the algebra* D, *before each element of any row.*

If we use the identity matrix as a left factor in multiplication with μ, with the 1 in the ith row and column replaced by $\lambda(\lambda \neq c)$, an element of the algebra D, we are merely prefixing λ as a factor to the elements of the ith row of the matrix μ. We shall call this altered matrix $e^{(i)}_{\lambda r}$, where (i) means that the ith 1 has been replaced by λ. It is by this transformation and the one following that we are able most easily to reduce the diagonal elements of the matrix to 1's, for

$$(d_{11}^{-1}\ d_{22}^{-1}\ d_{33}^{-1} \ldots\ldots\ d_{nn}^{-1})\ (d_{11}\ d_{22}\ d_{33} \ldots\ldots\ d_{nn}) = (1, 1, 1, \ldots\ldots 1_n),$$

where the $d_{ii} \neq 0, \qquad (i = 1 \ldots n).$

The inverse of λ is λ^{-1}. Then we may write the inverse of $e^{(i)}_{\lambda r}$ as $e^{(i)}_{\lambda^{-1} r}$. The product of these two in either order gives us the identity matrix, I.

(*v*) *The insertion of the same factor* ρ, *after each element of any column.*

If we use the identity matrix as a right factor in multiplication with μ, with the 1 in the jth row and column replaced by $\rho(\neq 0)$, an element of the algebra D, we are merely postfixing ρ as a factor to the elements of the jth column of the matrix μ.

Likewise, the newly re-arranged matrix, previously an identity matrix, may be called $e^{(j)}_{\rho_c}$ where the (j) means that the jth 1 has been replaced by ρ. ρ and ρ^{-1} are elements of D and inverses, one of the other. The inverse of e_{ρ_c} will be $e_{\rho_c^{-1}}$ since the products of the two in either order will be the identity matrix, or I.

We can readily define these matrices occurring in the elementary transformations to be non-singular since $\sum_{i=1}^{n} c_i d_{i,j} \neq 0$ and $\sum_{i=1}^{n} d_{i,j} b_j \neq 0$ for the c_i and the b_j, not all zero.

Defining a matrix μ, with elements in D, as having row-rank r_r, if it has τ left linearly independent rows, or as having column-rank rc' if it has σ right linearly independent columns we have

Theorem 1.—The row-rank and the column-rank of μ, with elements in D, are unaltered by transformations (1)...(5).

Under transformation (1) there are three cases to consider:—

Case I. $i \leq r_r$, $j \leq r_r$ $(i \neq j)$ $(r_r < n)$.

The addition to the elements of the ith row of the products of any element k, of the algebra D, into the corresponding elements of the jth row, k being used as a left factor, does not alter r_r, for the number of left linearly dependent rows remain the same. Hence r_r is not increased. r_r is not decreased because the ith row becomes a left combination of the ith row and k times the jth row. It is left linearly independent of any other row of μ.

Case II. $i < r_r$, $j > r_r$.

Here the number of left linearly dependent rows remain the same. r_r is not increased since the ith row becomes the left linear combination of the ith row with k times the jth row. It is left linearly independent of any other row of μ.

Case III. $i > r_r$, $j > r_r$ $(i \neq j)$.

The τ left linearly independent rows are not affected in this case. Hence the r_r is not decreased. The ith row is a left linear combination of the ith row and k times the jth row—both of which are left linearly dependent on a row or rows of the matrix. Then r_r is not increased.

The same holds for transformation (2) in dealing with σ right linearly independent columns with the exception that we substitute

right for *left*, *column* for *row*, and use k as a *right* rather than as a *left* factor.

It is obvious that (3) does not affect the left linear independence of the τ rows, or the right linear independence of the τ columns, since this transformation merely interchanges rows or columns, depending on the manner in which it is used as a factor.

In transformations (4) and (5) the λ and the ρ are non-zero elements chosen from the algebra D, their use does not alter the left linear independence of the τ rows or the right linear independence of the σ columns.

IV. *Reduction of Matrices over any Division Algebra to Diagonal Matrices.*

Consider the reduction of the square matrix μ of order n,

$$\mu=(d_{ij}), \qquad (i, j=1\ldots\ldots n).$$

We may assume $d_{11}\neq 0$; for, if it were zero we could interchange some column or row with the first column or row so that we could have the element in the (11) place not zero and by *Theorem 1*, at the same time not disturb the row—or the column-ranks.

Multiplying μ on the left by $e^{(i)}_{\lambda}$ where λ is d_{11}^{-1} we get μ_1, in which the first row is $1, \delta'_{1j}$ $(j=1\cdots\cdots n)$ and the other rows are d_{ij} $(i=2\ldots\ldots n;\ j=1\ldots\ldots n)$, $\delta'_{1j}=d^{-1}_{11}\ d^{1j}_{1j}$, and because of the property of closure (defined in *Algebras and their Arithmetics*, Art. 87) is an element of the division algebra D.

Using transformation (1) where p_1 is a series of products $\pi^{n}_{i=2}\ a^{(i2)}_{-k}$ $(i=2\ldots\ldots n)$ and $-k$ is of the form $-d_{i1}$ $(i=2\ldots\ldots n)$ we have the expression

$$\mu_2=\left(\prod^{n}_{i=2} a^{(i2)}_{-k}\right)\mu_1 \, *$$

where μ_2 has 1 in the (11) place, zeros in the ($i1$) places, $(i=2\ldots\ldots n)$ and δ_{kj} $(i, j, k=2\ldots\ldots n)$ elsewhere. Here $\delta^{(i)}_{kj}=d_{kj}-d_{k1}\,\delta'_{1j}$ $(i, j, k=2 \ldots\ldots n)$.

* Not independent of order.

Using transformation (2) where $q1$ is of the form $b^{(1j)}_{-k}$ $(j=2......n)$ and where $-k$ is of the form $-\delta'_{1j}$ $(j=2......n)$ we get

$$\mu_3=\mu_2\left(\prod_{j=2}^{n} b^{(1j)}_{-k}\right)^*$$

where μ_3 has 1 in the (11) place, zeros in the $(i1)$ and the $(1j)$ places $(i, j=2...n)$ and the remaining $(n-1)^2$ places have $\Delta^{(i)}_{jk}(i, j, k=2...n)$.

Here the $$\Delta^{(i)}_{jk}=\delta^{(i)}_{jk}-\delta'_{1k}\ (i, j, k=2...n),$$

and on account of the property of closure is an element of the division algebra D.

Taking the sub-matrix of order $(n-1)$ composed of the Δ's we can reduce it step by step, in the same manner until we reach the diagonal form known as the diagonal matrix, $(1, 1, 1,\cdots 1_r, 0, 0\cdots 0)$.

For example, taking the case of the non-singular matrix μ with elements in D, where $n=2$

$$\mu=\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22}\end{pmatrix}.$$

Form (4) and (5) we may select as a left factor

$$e^{(1)}_{\lambda_r}=\begin{pmatrix} d^{-1}_{11} & 0 \\ 0 & 1\end{pmatrix},\quad d_{11}=0;$$

and as a right facter

$$e_{\rho_c}=\begin{pmatrix} 1 & 0 \\ 0 & d^{-1}_{22}\end{pmatrix},\quad d_{22}\neq 0.$$

Then $$e^{(1)}_{\lambda_r}\ \mu\ e^{(2)}_{\rho_c}=\begin{pmatrix} d^{-1}_{11} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & d^{-1}_{22}\end{pmatrix},$$

$$=\begin{pmatrix} 1 & \delta_{12} \\ d_{21} & 1\end{pmatrix}=\mu_1\ ;$$

where $\delta_{12}=d^{-1}_{11}\ d_{12}\ d^{-1}_{22}$ is an element of the division algebra D, by property of closure.

By (1) and (2) we select

$$a^{(21)}_{-k_r}=\begin{pmatrix} 1 & 0 \\ -d_{21} & 1\end{pmatrix} \text{ and } b^{(12)}_{-k_c}=\begin{pmatrix} 1 & -\delta_{12} \\ 0 & 1\end{pmatrix}.$$

* Not independent of order.

Then $$a^{(21)}_{-k_r}\,\mu_1 b^{(12)}_{-k_c}=\begin{pmatrix}1 & 0\\ -d_{21} & 1\end{pmatrix}\begin{pmatrix}1 & \delta_{12}\\ d_{21} & 1\end{pmatrix}\begin{pmatrix}1 & -\delta_{12}\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}1 & 0\\ 0 & \delta_{22}\end{pmatrix}=\mu_2\,;$$

where $\delta_{22}=1-d_{21}\delta_{12}$ is an element of the division algebra, by the property of closure.

Using $e^{2}_{\rho_c}$ again with ρ_c as δ^{-1}_{22} $(\neq 0)$ we have

$$\mu_2 e^{(2)}_{\rho_c}=\begin{pmatrix}1 & 0\\ 0 & \delta_{22}\end{pmatrix}\begin{pmatrix}1 & 0^{-1}\\ 0 & \delta_{22}\end{pmatrix}=\mathrm{I}.$$

Then $p\mu q=\mathrm{I}$ where $p=a^{(21)}_{-k}\,e^{(1)}_{\lambda}$ and $q=e^{(1)}_{\rho}\,b^{(12)}_{-k}\,e^{(2)}_{\rho}$.

p and q are non-singular matrices since they are products of non-singular matrices occurring in the elementary transformations(1).........(5).

Multiplying on the left by p^{-1} and on the right by q^{-1} we get

$$p^{-1}p\mu qq^{-1}=p^{-1}\mathrm{I}q^{-1}=(p^{-1}q^{-1})=(qp)^{-1}=\mu.$$

So, by the process that we used to reduce μ to a diagonal matrix we can use the factor $(qp)^{-1}$ to pass again to μ.

Theorem 2.—If exactly r rows of the square matrix μ are left linearly independent with respect to D while all remaining rows are left linearly dependent on them, then exactly r columns of this matrix are, consequently, right linearly independent with respect to D while all remaining columns are right linearly dependent on them.

Proof follows by reducing the matrix to a diagonal matrix after application of *Theorem 1*. This diagonal matrix is square. Hence there will be $n-r$ rows and columns composed wholly of zeros. For, every row whose every element is zero there corresponds a column whose every element is zero. The converse is obvious.

Corollary.—The row-rank and the column-rank are equal.

This follows from the consideration of the diagonal matrix and *Theorem 2*.

Definition.—Then we may define the rank of a matrix to be the maximum number of rows left linearly independent with respect to D, or the maximum number of columns right linearly independent with respect to D.

Theorem 3.—If a matrix is non-singular, it has an inverse.

On reducing any matrix μ, with elements in D, to a normal form we see that

$$e_1 e_2 \ldots\ldots e_n \ \mu \epsilon_1 \epsilon_2 \epsilon_3 \ldots\ldots \epsilon_\nu = \text{a diagonal matrix},$$

where e_i and the ϵ_i are non-singular matrices occurring when we perform some of the elementary transformations. The products of the e_i on the left, may be expressed as a single matrix p. Likewise, the products of the ϵ_i, on the right, may be expressed as the matrix q. Then $p\mu q$=a diagonal matrix.

If we take μ to be equal to a non-singular matrix, the diagonal matrix becomes I, by *Theorem 1*. Since $p\mu q = 1$, $\mu = p^{-1}q^{-1}$ has the inverse qp. Hence every non-singular square matrix has an inverse, and at the same time we see that any non-singular matrix μ is a product of the generator matrices e_i and ϵ_i.

Theorem 4.—If a matrix μ, with elements in D, *has an inverse, it is non-singular.*

Choose p and q as in *Theorem 3* so that $p\mu q$ is a diagonal matrix δ. Suppose, contrary to the theorem, that μ is singular. Then δ has one or more rows of zeros. But

$$\mu\mu^{-1} = \mathrm{I}.$$

Hence $$p\mu q q^{-1} \mu^{-1} p^{-1} = \mathrm{I},$$

which is impossible since the product of δ by any matrix whatever has a row of zeros and is not equal to I. Hence every matrix having an inverse is non-singular.

Theorem 5.—(a) The product $\mu\nu$, of two non-singular matrices is non-singular.

(*b*) The product of a non-singular matrix μ by a singular matrix μ_1 is a singular matrix μ_2.

(*a*) For, μ and ν have inverses by *Theorem 4*, and hence $\mu\nu$ has inverse $\nu^{-1}\mu^{-1}$. Then by *Theorem 3*, the product $\mu\nu$ is non-singular.

(*b*) In the product, $\mu\mu_1 = \mu_2$, we are able to multiply on the left by μ^{-1} getting $\mu_1 = \mu^{-1}\mu_2$. Suppose μ_2 non-singular, then we would have the product of two non-singular matrices equal to a singular matrix. This is a contradiction of the first part of the theorem. Hence μ_2 is singular.

Theorem 6.—If $\mu=\mu_1\mu_2$ is a product of two matrices μ_1 and μ_2, the rank of the product matrix μ cannot exceed the rank of either of the factor matrices, μ_1 and μ_2.

First, let r be the rank of μ_1. If $r=n$, the rank of μ cannot exceed n. If $r<n$, there is a left linear dependence between every $r+1$ rows of μ_1, and so here will be a left linear dependence between every $r+1$ rows of μ. Thus μ cannot have $r+1$ left linearly independent rows. Hence it follows that the rank of μ cannot exceed r.

Next, let r' be the rank of μ_2. If $r'=n$, the rank of μ cannot exceed r', for, ij cannot exceed n. If $r'<n$ there is a right linear dependence between every $r'+1$ columns of μ_2 and therefore there is a right linear dependence between $r'+1$ columns of μ. Thus μ cannot have $r'+1$ right linear independent columns. Hence the rank of μ cannot exceed r.

It may be observed that it is possible for the rank of μ to be less than the ranks of both μ_1 and μ_2.

Theorem 7.—The rank of a matrix is unaltered when multiplied by a non-singular square matrix of the same order as a right or as a left factor.

Let μ' be any square matrix of order n and of rank r $(r<n)$, μ_1, a non-singular square matrix of order n, and μ'', a product matrix of order n. We are to show that the rank of μ'' equals that of μ'.

First, consider the equality $\mu_1\mu'=\mu''$. Multiplying both sides on the left by μ_1^{-1} we get $\mu'=\mu_1^{-1}\mu''$. By *Theorem 6*, the rank of μ'' cannot exceed the rank of μ' and the rank of μ' cannot exceed that of μ''. Hence the rank of μ'' is equal to the rank of μ'.

In the product $\mu'\mu_2=\mu''$, where μ_2 is non-singular and μ', μ'' are the same as before, we multiply both sides on the right by μ_2^{-1} and get $\mu'=\mu''\mu_2^{-1}$. The rank of μ'' cannot exceed that of μ' and and the rank of μ' cannot exceed that of μ''. Hence the rank of μ'' equals that of μ'.

Corollary.—The rank of a matrix is unchanged when multiplied by other two non-singular matrices of the same order as a left and as a right factor.

Suppose μ is of rank r, μ_1 and μ_2 are non-singular,—all of order n, with elements in the division algebra D. We are to show that in the product $\mu_1\mu\mu_2=\mu'$, μ' has the same rank as μ. By association

$$(\mu_1\mu)\mu_2=\mu_1(\mu\mu_2)=\mu'.$$

By *Theorem 7* we see that $(\mu_1\mu)$ and $(\mu\mu_2)$ each have rank r. Therefore, the rank of μ' is equal to that of μ.

V. *Equivalence of Matrices.*

We may say that two matrices μ ond μ', with elements in the algebra D, have row-equivalence in D when they have the same number of columns, and when every row of μ is left linearly dependent with respect to D on the rows of μ', and when every row of μ' is left linearly dependent with respect to D on the rows of μ. Thus the matrices μ and μ' are row-equivalent in D when and only when there exist relations of both the forms

(1) $\mu = p\mu'$ $\qquad\qquad$ $\mu' = r\mu$

where p, r are products of non-singular square generator matrices occurring in the elementary transformations (1)...(5) having the same order as μ, μ' and with elements in the same algebra D. From *Theorem 7* we saw that these two matrices, μ and μ', have the same rank. Hence they have the same mumber of left linearly independent rows. Therefore, equivalent in D.

Again, any two matrices, $\mu\mu$ and μ', with elements in D, will be said to have column-equivalence in D when they have the same number of rows and when every column of μ is right linearly dependent with respect to D on the colums of μ', and every column of μ' is right linearly dependent with respect to D on the columns of μ. Thus μ and μ' are column equivalent when and only when there exist relations of both the forms

(2) $\mu = \mu' q$; $\qquad\qquad$ $\mu' = \mu s$,

where q, s are non-singular square matrices of the same order as μ, μ' and with elements in the same algebra D. By *Theorem 7* the two matrices, μ and μ', have the same rank. Hence the same number of right linearly independent columns. Therefore, equivalent in D.

Thus the matrices, μ and μ', are called equivalent in D when and only when there exist relations of both the forms

(3) $\mu = p\mu' q$, $\qquad\qquad$ $\mu' = r\mu s$.

From corollary following *Theorem 7* we see that these two matrices μ and μ', have the same rank, *i.e.*, the same number of right linearly independent columns. Then μ is said to be equivalent to μ' or μ' is said to be equivalent to μ. We denote this relation by

I. $\mu \smile \mu'$ $\qquad$ or $\qquad$ $\mu' \smile \mu$

This is known as the first principle of equivalence. The other two follow :—

II. *Every matrix μ is self-equivalent.*

p and q may be the products of identity matrices in which case $p\mu q = \mu$ or $\mu \smile \mu$.

III. *If two matrices, μ and μ', are each equivalent to a third matrix, μ'', they are equivalent to each other.*

This is known as the principle of transitivity. Let

$$p_1\mu q_1=\mu'',$$

$$r_1\mu' s_1=\mu''.$$

Then
$$p_1\mu q_1=r_1\mu' s_1,$$

$$r_1^{-1}p_1\mu q_1 s_1^{-1}=\mu',$$

or
$$p\mu q=\mu' ;$$

where $p=r_1^{-1}p_1$ and $q=q_1 s_1^{-1}$ are non-singular square matrices with elements in the algebra D, since r_1, p_1, q_1, s_1 were matrices with like elements. Hence, if $\mu \backsim \mu''$ and $\mu' \backsim \mu''$, it follows that $\mu \backsim \mu'$.

Theorem 8.—Equivalent matrices, μ and μ', with element in D, *have equal rank.*

In corollary following *Theorem 7* we have shown that in the relation $p\mu q=\mu'$ the ranks of μ and μ' are the same. We have shown that $\mu \backsim \mu'$ if such a relation holds. Hence the proof.

The preceding theorem leads to the converse :—

Theorem 9.—Matrices, μ and μ', having equal rank, are equivalent, in D.

From the article on the reduction of matrices we see that matrices of rank r $(r \leq n)$, whose elements are elements of the algebra D, can easily be put into the form of diagonal matrices. We have defined μ and μ' to be equivalent if the relation $p\mu q=\mu'$ holds. In the process of reduction $p_1\mu q_1 \backsim (1, 1, 1, \ldots 1_r, 0, \ldots 0_n)$, and $p_2\mu' q_2 \backsim (1, 1, 1, \ldots 1_r, 0, \ldots 0_n)$, since by hypothesis, they have the same rank. Then, by the third principle of equivalence

$$p_2\mu' q_2 \backsim p_1\mu q_1$$

and
$$\mu' \backsim p_2^{-1}p_1\mu q_1 q_2^{-1}$$

or
$$\mu' \backsim p\mu q$$

in which $p=p_2^{-1}p_1$, $q=q_1 q_2^{-1}$ where the p's and the q's are each non-singular products of non-singular matrices occurring in the elementary transformations (1)...(5).

On a type of solution of Einstein's gravitational equations.

BY

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(*Calcutta*)

1. The gravitational field in vacuum is according to Einstein, given by the differential equations.

$$G_{\mu\nu}=0, \qquad (\mu,\nu=1,\ 2,\ 3,\ 4)$$

The solutions of these equations studied hitherto are not large in number. The most well known are those* with a centre and an axis of symmetry. A solution with singularities on an axis has been given by H. E. J. Curzon.† The object of the present paper is to study some particular types of solutions of the above equations. They have been obtained by direct integration on the assumption of certain form for the line element. Some new informations are obtainable in this way in certain cases. For instance, it can be proved that there is only one solution of the type

$$ds^2=f_1(x_1)dx_1^2+f_2(x_1)dx_2^2+f_3(x_1,\ x_2)dx_3^2-f_4(x_1)dx_4^2 \quad \ldots \quad (\text{I})$$

which is really identical with the Schwarzschild's solution with a single singularity.

2. Let us assume the following form for the line element,

$$ds^2=a_1g_{11}dx_1^2+a_2g_{22}dx_2^2+a_3g_{33}dx_3^2+a_4g_{44}dx_4^2,$$

where g's are functions of $x_1,\ x_2,\ x_3,\ x_4,$ and $a_1,\ a_2,\ a_3,\ a_4$ are numerical constants introduced for manipulating the signs of the functions

* K. Schwarzschild, Berl. Ber. 1916
T. Droste, Amst. Versl. 25 (1916)
H. Weyl, Ann. d. Phys. 54 (1917)
T. Levi-Civita, Rend. Acc. Linc. (5) 26 (1917)

† Proc. Lond. Math. Soc.; Series 2, vol. 23, 1924.

involved. We shall first calculate the functions $G_{\mu\nu}$ and put them in symmetrical forms which would greatly simplify our subsequent analysis.

We have

$$G_{\mu\nu} = -\frac{\partial}{\partial x_\alpha}\{\mu\nu,\alpha\} + \{\mu\alpha,\beta\}\{\nu\beta,\alpha\}$$

$$+\frac{\partial^2}{\partial x_\mu\,\partial x_\nu}\log\sqrt{-g} - \{\mu\nu,\alpha\}\frac{\partial}{\partial x_\alpha}\log\sqrt{-g};$$

Here

$$\{\mu\mu,\ \mu\} = \tfrac{1}{2}\frac{\partial}{\partial x_\mu}\log(g_{\mu\mu}),$$

$$\{\mu\nu,\ \mu\} = \tfrac{1}{2}\frac{\partial}{\partial x_\nu}\log(g_{\mu\mu}),$$

$$\{\mu\mu,\nu\} = -\frac{a_\mu\ g_{\mu\mu}}{2a_\nu\ g_{\nu\nu}}\frac{\partial}{\partial x_\nu}\log(g_{\mu\mu}),$$

$$\{\mu\nu,\sigma\} = 0.$$

Put

$$g_{\mu\mu} = e^{2h_{\mu\mu}}$$

then

$$\{\mu\mu,\ \mu\} = \frac{\partial}{\partial x_\mu}h_{\mu\mu},$$

$$\{\mu\nu,\mu\} = \frac{\partial}{\partial x_\nu}h_{\mu\mu},$$

$$\{\mu\mu,\nu\} = -\binom{\mu}{\nu}e^{2(h_{\mu\mu}-h_{\nu\nu})}\frac{\partial}{\partial x_\nu}h_{\mu\mu},$$

$$\{\mu\nu,\ \sigma\} = 0,$$

where $\binom{\mu}{\nu}$ denotes $\dfrac{a_\mu}{a_\nu}$.

Let the four values through which α, μ, ν run be denoted, for the sake of convenience, by m, n, r, p,

then $-\dfrac{\partial}{\partial x_\alpha}\{\mu\nu, \alpha\}$ in the expression for G_{pp} stands for

$$\binom{p}{m}\frac{\partial}{\partial x_m}\left\{e^{2(h_{pp}-h_{mm})}\frac{\partial}{\partial x_m}h_{pp}\right\}$$

$$+\binom{p}{n}\frac{\partial}{\partial x_n}\left\{e^{2(h_{pp}-h_{nn})}\frac{\partial}{\partial x_n}h_{pp}\right\}$$

$$+\binom{p}{r}\frac{\partial}{\partial x_r}\left\{e^{2(h_{pp}-h_{rr})}\frac{\partial}{\partial x_r}h_{pp}\right\}$$

$$-\frac{\partial^2}{\partial x_p^2}h_{pp}.$$

or

$$\sum_\lambda^{(m,\, n,\, r)}\binom{p}{\lambda}e^{2(h_{pp}-h_{\lambda\lambda})}\left\{\frac{\partial^2}{\partial x^2_\lambda}h_{pp}\right.$$

$$\left.+2\left(\frac{\partial}{\partial x_\lambda}h_{pp}\right)^2-2\frac{\partial}{\partial x_\lambda}h_{pp}\frac{\partial}{\partial x_\lambda}h_{\lambda\lambda}\right\}$$

$$-\frac{\partial^2}{\partial x^2_p}h_{pp}.$$

Similarly $\{\mu\alpha, \beta\}\{\nu\beta, \alpha\}$ in G_{pp} may be expressed as

$$\sum_\lambda^{(m,\, n,\, r,\, p)}\left(\frac{\partial}{\partial x_p}h_{\lambda\lambda}\right)^2$$

$$-\sum_\lambda^{(m,\, n,\, r)}2\binom{p}{\lambda}e^{2(h_{pp}-h_{\lambda\lambda})}\left(\frac{\partial}{\partial x_\lambda}h_{pp}\right)^2;$$

and $\dfrac{\partial^2}{\partial x_\mu\partial x_\nu}\log\sqrt{-g}$ in G_{pp} may be put as

$\dfrac{\partial^2}{\partial x_p^2}h$, where h denotes $h_{mm}+h_{nn}+h_{rr}+h_{pp}$;

also $\quad -\{\mu, \nu, a\} \dfrac{\partial}{\partial x_a} \log\sqrt{-g}$ in G_{pp} may be put in the form

$$\sum_{\lambda}^{(m,n,r)} \binom{p}{\lambda} e^{2(h_{pp}-h_{\lambda\lambda})} \frac{\partial}{\partial x_\lambda} h \frac{\partial}{\partial x_\lambda} h_{pp}$$

$$- \frac{\partial}{\partial x_p} h_{pp} \frac{\partial}{\partial x_p} h.$$

Hence G_{pp} comes out after simplification in the form

$$\sum_{\lambda}^{(m,n,r)} \left[\binom{p}{\lambda} e^{2(h_{pp}-h_{\lambda\lambda})} \left\{ \frac{\partial^2}{\partial x_\lambda^2} h_{pp} \right.\right.$$

$$\left. + \frac{\partial}{\partial x_\lambda} h_{pp} \frac{\partial}{\partial x_\lambda} (h - 2h_{\lambda\lambda}) \right\}$$

$$\left. + \frac{\partial^2}{\partial x_p^2} h_{\lambda\lambda} + \left(\frac{\partial}{\partial x_p} h_{\lambda\lambda} \right)^2 - \frac{\partial}{\partial x_p} h_{pp} \frac{\partial}{\partial x_p} h_{\lambda\lambda} \right] \quad \ldots \quad \text{(A)}$$

Again

$$- \frac{\partial}{\partial x_a} \{\mu\nu, a\} \quad \text{in} \quad G_{mn}$$

$$= - \frac{\partial}{\partial x_m} \{mn, m\} - \frac{\partial}{\partial x_n} \{mn, n\}$$

$$= - \frac{\partial^2}{\partial x_m \partial x_n} (h_{mm} + h_{nn})\,;$$

Similarly $\{\mu a, \beta\}\{\nu\beta, a\}$ in G_{mn}

$$= \sum_{\lambda}^{(m,n,r,p)} \frac{\partial}{\partial x_m} h_{\lambda\lambda} \frac{\partial}{\partial x_n} h_{\lambda\lambda} + 2 \frac{\partial}{\partial x_n} h_{mm} \frac{\partial}{\partial x_m} h_{nn}\,;$$

and $\quad \dfrac{\partial^2}{\partial x_\mu \partial x_\nu} \log\sqrt{-g}$ in G_{mn}

$$= \frac{\partial^2}{\partial x_m \partial x_n} h \ldots$$

also $$-\{\mu\nu, a\}\frac{\partial}{\partial x_a}\log\sqrt{-g} \quad \text{in} \quad G_{mn}$$

$$=-\frac{\partial}{\partial x_n}h_{mm}\frac{\partial}{\partial x_m}h-\frac{\partial}{\partial x_m}h_{nn}\frac{\partial}{\partial x_n}h;$$

Hence $G_{mn(m\neq n)}$ comes out after simplification in the form

$$\sum_{\lambda}^{(r,p)}\left[\frac{\partial^2}{\partial x_m \partial x_n}h_{\lambda\lambda}+\frac{\partial}{\partial x_m}h_{\lambda\lambda}\frac{\partial}{\partial x_n}h_{\lambda\lambda}\right.$$

$$\left.-\frac{\partial}{\partial x_n}h_{mm}\frac{\partial}{\partial x_m}h_{\lambda\lambda}-\frac{\partial}{\partial x_m}h_{nn}\frac{\partial}{\partial x_n}h_{\lambda\lambda}\right] \quad \dots \quad \text{(B)}$$

3. The above forms (A) and (B) of $G_{\mu\nu}$ correspond to the line element of the type

$$ds^2=a_1e^{2h_{11}}dx_1^2+a_2e^{2h_{22}}dx_2^2+a_3e^{2h_{33}}dx_3^2+a_4e^{2h_{44}}dx_4^2$$

We shall consider in this paper the following particular case. (I)

Let h_{11}, h_{22}, h_{44} be functions of x_1 only while h_{33} is a function of x_1 and x_2. Also let $a_1=a_2=a_3=-a_4=-1$; Then we have the following five differential equations to solve.

$$G_{11}\equiv\frac{\partial^2}{\partial x_1^2}(h_{22}+h_{33}+h_{44})+\left(\frac{\partial}{\partial x_1}h_{22}\right)^2+\left(\frac{\partial}{\partial x_1}h_{33}\right)^2$$

$$+\left(\frac{\partial}{\partial x_1}h_{44}\right)^2-\frac{\partial}{\partial x_1}h_{11}\frac{\partial}{\partial x_1}(h_{22}+h_{33}+h_{44})=0$$

$$G_{22}\equiv e^{2(h_{22}-h_{11})}\left\{\frac{\partial^2}{\partial x_1^2}h_{22}+\frac{\partial}{\partial x_1}h_{22}\right.$$

$$\left.\dots\frac{\partial}{\partial x_1}(h_{22}+h_{33}+h_{44}-h_{11})\right\}+\frac{\partial^2}{\partial x_2^2}h_{33}+\left(\frac{\partial}{\partial x_2}h_{33}\right)^2=0$$

$$G_{33}\equiv e^{2(h_{33}-h_{11})}\left\{\frac{\partial^2}{\partial x_1^2}h_{33}+\frac{\partial}{\partial x_1}h_{33}\frac{\partial}{\partial x_1}(h_{22}+h_{33}+h_{44}-h_{11})\right.$$

$$\left.+e^{2(h_{33}-h_{22})}\left\{\frac{\partial^2}{\partial x_2^2}h_{33}+\left(\frac{\partial}{\partial x_2}h_{33}\right)^2\right\}\right\}=0$$

$$G_{44}\equiv -e^{2(h_{44}-h_{11})}\left\{\frac{\partial^2}{\partial x_1^2}h_{44}\right.$$

$$\left.+\frac{\partial}{\partial x_1}h_{44}\frac{\partial}{\partial x_1}(h_{22}+h_{33}+h_{44}-h_{11})\right\}=0$$

$$G_{12}\equiv\frac{\partial^2}{\partial x_1\partial x_2}h_{33}+\frac{\partial}{\partial x_1}h_{33}\frac{\partial}{\partial x_2}h_{33}-\frac{\partial}{\partial x_1}h_{22}\frac{\partial}{\partial x_2}h_{33}=0$$

Multiplying G_{22} by $e^{2(h_{33}-h_{22})}$ and subtracting G_{33} from this, we get—

$$\frac{\partial^2}{\partial x_1^2}(h_{22}-h_{33})+\left(\frac{\partial}{\partial x_1}h_{22}\right)^2-\left(\frac{\partial}{\partial x_1}h_{33}\right)^2$$

$$+\frac{\partial}{\partial x_1}(h_{44}-h_{11})\frac{\partial}{\partial x_1}(h_{22}-h_{33})=0 \qquad \ldots \qquad \ldots \quad (1)$$

Simplifying G_{11} by means of G_{44} we have also

$$\frac{\partial^2}{\partial x_1^2}(h_{22}+h_{33})+\left(\frac{\partial}{\partial x_1}h_{22}\right)^2+\left(\frac{\partial}{\partial x_1}h_{33}\right)^2$$

$$-\frac{\partial}{\partial x_1}(h_{44}+h_{11})\frac{\partial}{\partial x_1}(h_{22}+h_{33})=0 \quad \ldots \quad (2)$$

The equations (1) and (2) are equivalent to

$$\frac{\partial^2}{\partial x_1^2}h_{22}+\left(\frac{\partial}{\partial x_1}h_{22}\right)^2=\frac{\partial}{\partial x_1}h_{11}\frac{\partial}{\partial x_1}h_{22}$$

$$+\frac{\partial}{\partial x_1}h_{33}\frac{\partial}{\partial x_1}h_{44} \qquad \ldots \quad (3)$$

$$\frac{\partial^2}{\partial x_1^2} h_{33} + \left(\frac{\partial}{\partial x_1} h_{33} \right)^2 = \frac{\partial}{\partial x_1} h_{11} \frac{\partial}{\partial x_1} h_{33}$$

$$+ \frac{\partial}{\partial x_1} h_{33} \frac{\partial}{\partial x_1} h_{44} \quad \ldots \quad (4)$$

From the equation (3) it can at once be seen that $\frac{\partial}{\partial x_1} h_{33}$ cannot involve x_2; or in other words h_{33} must have the form

$$\phi_1(x_1) + \phi_2(x_2).$$

Since $\frac{\partial^2}{\partial x_1 \partial x_2} h_{33} = 0$, the equation G_{12} reduces to

$$\frac{\partial}{\partial x_1} h_{33} = \frac{\partial}{\partial x_1} h_{22} \quad \ldots \quad (5)$$

Eliminating $\frac{\partial}{\partial x_1} h_{33}$ by means of (5) we have now the following three equations to consider:—

$$\frac{\partial^2}{\partial x_1^2} h_{22} + \left(\frac{\partial}{\partial x_1} h_{22} \right)^2 = \frac{\partial}{\partial x_1} h_{22} \frac{\partial}{\partial x_1} (h_{11} + h_{44}) \quad \ldots \quad (6)$$

$$\frac{\partial^2}{\partial x_1^2} h_{44} + \left(\frac{\partial}{\partial x_1} h_{44} \right)^2 = \frac{\partial}{\partial x_1} h_{44} \frac{\partial}{\partial x_1} (h_{11} - 2h_{22}) \quad \ldots \quad (7)$$

$$\frac{\partial^2}{\partial x_2^2} h_{33} + \left(\frac{\partial}{\partial x_2} h_{33} \right)^2 + e^{2(h_{22} - h_{11})}$$

$$\times \left\{ \frac{\partial}{\partial x_1} h_{22} \frac{\partial}{\partial x_1} (h_{22} + 2h_{44}) \right\} = 0 \quad \ldots \quad (8)$$

Put $\frac{\partial}{\partial x_1} h_{\mu\mu} = H_{\mu\mu}$ and $\frac{\partial}{\partial x_2} h_{\mu\mu} = K_{\mu\mu}$;

then (5) may be written as

$$H_{33} = H_{22} \quad \ldots \quad (5')$$

(6) may be written as

$$\frac{\partial}{\partial x_1}\log H_{22}+H_{22}=H_{11}+H_{44} \qquad \ldots \quad (6')$$

(7) may be written as

$$\frac{\partial}{\partial x_1}\log H_{44}+H_{44}=H_{11}-2H_{22} \qquad \ldots \quad (7')$$

(8) may be written as

$$\frac{\partial}{\partial x_2}K_{33}+(K_{33})^2+e^{2(h_{22}-h_{11})}$$

$$\times\{H_{22}(H_{22}+2H_{44})\}=0 \qquad \ldots \quad (8')$$

From (6′) and (7′) we have

$$\frac{\partial}{\partial x_1}\log\frac{H_{22}}{H_{44}}=2H_{44}+H_{22} \qquad \ldots \quad (9)$$

Let us represent $\frac{H_{22}}{H_{44}}$ by M, then (9) becomes

$$\frac{\partial}{\partial x_1}\log M=(2+M)H_{44}$$

Whence $H_{44}=\frac{1}{M(2+M)}\frac{\partial M}{\partial x_1}$ ... (10)

Consequently $H_{22}=\frac{1}{2+M}\frac{\partial M}{\partial x_1}$ ... (11)

and from (7′),

$$H_{11}=\frac{\partial}{\partial x_1}\log H_{44}+\frac{1+2M}{M(2+M)}\frac{\partial M}{\partial x_1} \qquad \ldots \quad (12)$$

Integrating (10) we have

$$h_{44}=\tfrac{1}{2}\log \frac{M}{2+M}c_4, \quad \text{where } c_4 \text{ is an arbitrary constant.}$$

Integrating (11) we have

$$h_{22}=\log(2+M)c_2, \quad \text{where } c_2 \text{ is an arbitrary constant.}$$

Integrating (12) we have

$$h_{11}=\log H_{44}+\log M^{\frac{1}{2}}(2+M)^{\frac{3}{2}}c_1, \quad \text{where } c_1 \text{ is an arbitrary constant}$$

$$=\log\left(\frac{2+M}{M}\right)^{\frac{1}{2}}\frac{\partial M}{\partial x_1}c_1$$

Since $H_{33}=H_{22}$, the part of h_{33} involving x_1.

i.e., $\phi_1(x_1)=\log(2+M)c_3$, where c_3 is an arbitrary constant.

In the above solutions M is evidently an arbitrary function of x_1.

To find the part of h_{33} involving x_2 let us revert to the equation (8′).

It can be shown that

$$e^{2(h_{22}-h_{11})}\{H_{22}(H_{22}+2H_{44})\}=\frac{c_2^2}{c_1^2}$$

Hence equation (8′) reduces to—

$$\frac{\partial}{\partial x_2}K_{33}+(K_{33})^2+\left(\frac{c_2}{c_1}\right)^2=0$$

Integrating this, the part of h_{33} involving x_2 *i.e.*, $\phi_2(x_2)$ has the form

$$\log b_1 \cos\left\{(x_2+b_2)\frac{c_2}{c_1}\right\}$$

where b_1 and b_2 are arbitrary constants.

Hence the required form for the line element is

$$ds^2=-\frac{2+M}{M}c_1^2dM_1^2-(2+M)^2c_2^2dx_2^2$$

$$-(2+M)^2c_3^2\cos^2\left\{(x_2+b_2)\frac{c_2}{c_1}\right\}dx_3^2+\frac{M}{2+M}c_4dx_4^2$$

By means of the following equations of transformation,

$$(2+M)c_1 = M'$$

$$(x_2+b_2)\frac{c_2}{c_1} + \frac{\pi}{2} = x'_2$$

$$\frac{c_3}{c_1}x_3 = x_3'$$

$$c_4 x_4 = x'_4$$

the above readily reduces to the form—

$$ds^2 = -\frac{1}{1-\frac{2c_1}{x_1}}dx_1^2 - x_1^2 dx_2^2 - x_1^2 \sin^2 x_2 dx_3^2 + \left(1-\frac{2c_1}{x_1}\right)dx_4^2$$

which is identical with Schwarzschild's solution.

Hence Einstein's equations admit of only one solution of the type (I) and this solution has one singularity and corresponds to a field with a centre of symmetry.

A study of some more solutions will be taken up later on.

In conclusion I wish to express my thanks to Prof. N. R. Sen at whose suggestion I took up the problem and under whose guidance I carried on the above investigation.

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3

ON DISCONTINUOUS FUNCTIONS WHOSE PROGRESSIVE DIFFERENTIAL COEFFICIENTS EXIST AT EVERY POINT.

BY

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In the present paper I have constructed some examples of discontinuous functions which possess at each point in the interval (0, 1) a right-hand differential co-efficient which is equal to zero and which are not constants in (0, 1). The first of these examples *viz.*, $f_1(x)$, possesses the following properties:—

(1) It is monotone and non-diminishing and hence of bounded variation.

(2) It is everywhere continuous on the right, *but has discontinuties of the first kind at an enumerable everywhere dense set.*

(3) At all the points in (0, 1)

$$D^+f_1(x)=D_+f_1(x)=0.$$

The second example, $f_2(x)$, has the following properties:—

(1) It is constant in each of the black intervals of a perfect set, but varies in the interval (0, 1).

(2) It is everywhere continuous on the right, *but has discontinuities of the first kind at an enumerable non-dense set.*

(3) It is of bounded variation.

(4) At all points in (0, 1)

$$D^+f_2(x)=D_+f_2(x)=0.$$

Further it may be noted that the functions $f_1(x)$ and $f_2(x)$ are such that

$$D^+f_1(x)=D_+f_1(x)=D^+f_2(x)=D_+f_2(x)=0,$$

everywhere, but

$$f_1(x) \neq f_2(x)^*$$

I take this opportunity to express my best thanks to Dr. Ganesh Prasad under whose guidance I have carried out these investigations.

Example I. Let the points of the interval (0, 1) be expressed in the dyad scale, so that

$$x = \frac{c_1}{2} + \frac{c_2}{2^2} + \frac{c_3}{2^3} + \dots + \frac{c_n}{2^n} + \dots ,$$

where each c is either zero or one.

For values of x not representable as ending decimals, let

$$y = f(x) = \frac{2c_1}{3} + \frac{2c_2}{3^2} + \dots + \frac{2c_n}{3^n} + \dots$$

For values of x representable as an ending decimal

$$x = \frac{c_1}{2} + \frac{c_2}{2^2} + \dots + \frac{1}{2^n},$$

let

$$y = f(x) = \frac{2c_1}{3} + \frac{2c_2}{3^2} + \dots + \frac{2}{3^n}.$$

(1) It is easy to see that if

$$X = \frac{C_1}{2} + \frac{C_2}{2^2} + \dots + \frac{C_n}{2^n} + \dots$$

be greater than

$$x = \frac{c_1}{2} + \frac{c_2}{2^2} + \dots + \frac{c_n}{2^n} \dots$$

there is a number r such that

$$c_r = 0, \quad \text{while } C_r = 1,$$

* Hahn (*Monartsh. F. Math.*, *Vol.* XVI, p. 161, 1905) has given two functions which have the same derivates, and their difference is not constant. But there are points where these derivates have infinite values. (In this connection see, S. Ruziewiez, "On the functions which have the same derivates and of which the difference is not constant," *Fundamenta Math.* Vol. I, pp. 148-51). In my examples the right hand derivatives of $f_1(x)$ and $f_2(x)$ are finite and equal.

and for all values of $n < r$

$$c_n = C_n.$$

$$\therefore \qquad Y = f_1(X) > y = f_1(x),$$

for, Y and y agree up to $(r-1)$ places, while at the rth place there is a 2 in Y and a 0 in y.

$f(x)$ is, therefore, monotone and non-diminishing and hence of bounded variation.

(2) The function $f(x)$ is continuous at all the points in $(0, 1)$ which are not representable as ending decimals in the scale of 2, for if x and X agree up to the nth place, y and Y also agree up to the nth place. It is also easy to see that at a point x which is representable as an ending decimal

$$x = \frac{c_1}{2} + \frac{c_2}{2^2} + \frac{c_3}{2^3} + \ldots + \frac{c_r}{2^r} + \frac{1}{2^{r+1}},$$

$$f(x+o) = f(x) = \frac{2c_1}{3} + \frac{2c_2}{3^2} + \ldots + \frac{2c_r}{3^r} + \frac{2}{3^{r+1}};$$

while

$$f(x-o) = \frac{2c_1}{3} + \frac{2c_2}{3^2} + \ldots + \frac{2c_r}{3^r} + \frac{1}{3^{r+1}}.$$

At all such points, therefore, $f(x)$ is continuous on the right, whilst it has got a discontinuity of the first kind on left. The set formed by these points (ending decimals) is evidently enumerable and everywhere dense.

(3) To prove that the function has a right-hand derivative which exists everywhere and is equal to zero, we consider the non-ending and the ending decimals separately:

(a) Let x be a point which is not representable as an ending decimal, *i.e.*, let

$$x = \frac{c_1}{2} + \frac{c_2}{2^2} + \ldots + \frac{c_n}{2^n} + \frac{c_{n+1}}{2^{n+1}} + \ldots \quad ,$$

so that from and after some place (mth, say) all the c's are not equal to 1. Then, the greatest number of 1's occurring one after the other in the representation of x must be finite $(=r$, say).

$$\text{Let } h_p = \frac{1}{2^p} + \frac{K_{p+1}}{2^{p+1}} + \frac{K_{p+2}}{2^{p+2}} + \ldots$$

and $$(x+h_p)=\frac{c_1}{2}+\frac{c_2}{2^2}+\ldots+\frac{c'_k}{2^k}+\frac{c'_{k+1}}{2^{k+1}}+\ldots,$$

then $$f(x)=\frac{2c_1}{3}+\frac{2c_2}{3^2}+\ldots+\frac{2c_k}{3^k}+\frac{2c_{k+1}}{3^{k+1}}+\ldots,$$

and $$f(x+h_p)=\frac{2c_1}{3}+\frac{2c_2}{3^2}+\ldots+\frac{2c'_k}{3^k}+\frac{2c'_{k+1}}{3^{k+1}}+\ldots,$$

so that $$f(x+h_p)-f(x)\leq\frac{1}{3^{k-1}}.$$

Also $$h_p\geq\frac{1}{2^p}.$$

$$\therefore\qquad \frac{f(x+h_p)-f(x)}{h_p}\leq\frac{\frac{1}{3^{k-1}}}{\frac{1}{2^p}}.$$

But $(p-k+1)$ is always finite and less than or equal to $r+1$, so that
$$(k-1)\geq(p-r-1).$$

Hence $$\lim_{h_p\to 0}\frac{f(x+h_p)-f(x)}{h_p}\leq\lim_{p\to\infty}\frac{2^p}{3^{p-r-1}},\leq 0.$$

But the incrementary ratio is always positive, because the function $f(x)$ is non-diminishing.

Therefore, the right-hand derivative exists at all points which are not representable as ending decimals in the scale of 2, and is equal to zero.

(*b*) Let x be representable as an ending decimal
$$x=\frac{c_1}{2}+\frac{c_2}{2^2}+\frac{c_3}{2^3}+\ldots+\frac{c_r}{2^r}+\frac{1}{2^{r+1}},$$

and let $$h_m=\frac{1}{2^m}+\frac{k_{m+1}}{2^{m+1}}+\frac{k_{m+2}}{2^{m+2}}+\ldots;$$

then, $$f(x)=\frac{2c_1}{3}+\frac{2c_2}{3^2}+\ldots+\frac{2c_r}{3^r}+\frac{2}{3^{r+1}}.$$

and $$f(x+h_m)=\frac{2c_1}{3}+\frac{2c_2}{3^2}+\ldots+\frac{2}{3^{r+1}}+\frac{0}{3^{r+2}}+\ldots+\frac{2}{3^m}+\frac{2k_{m+1}}{3^{m+1}}+\ldots;$$

so that

$$f(x+h_m)-f(x) \leq \frac{1}{3^{m-1}}.$$

$$\therefore \quad \frac{f(x+h_m)-f(x)}{h_m} \leq \frac{2^m}{3^{m-1}}.$$

Hence $$\lim_{h_m \to 0} \frac{f(x+h_m)-f(x)}{h_m} \leq \lim_{m \to \infty} \frac{2^m}{3^{m-1}} \leq 0.$$

As the incrementary ratio is always positive, there exists a right hand derivative equal to zero.

We have thus shown that at all the points in (0, 1)

$$D^+f(x)=D_+f(x)=0.$$

Example II. Consider Cantor's perfect set and its black intervals. When x lies in a black interval of the set, let $f(x)$ equal the square of the length of the black interval. When x is a point of the perfect set let $f(x)=f(x+0)$.

The function is evidently continuous at all the points complementary to the perfect set. At all the points that belong to the perfect set and are limiting points on both sides $f(x)$ is continuous and has the value zero, for we can always find an interval enclosing such a point and such that it does not contain any interval of the rth stage, where r is less than or equal to a given arbitrarily large number n. At all the points which are end points of a black interval, $f(x)$ is continuous on the right only, whilst it has got a discontinuity of the first kind on the left.

It is also easy to see that the total variation of the function cannot exceed

$$\frac{2}{3^2}+\frac{2^2}{3^4}+\frac{2^3}{3^6}+\ldots$$

$f(x)$ is, therefore, of bounded variation.

It is also evident that the right-hand derivative at all the points that are interior to, or are left-hand end points of black intervals, exists and is equal to zero. The remaining points are points of the set that are either limiting points on both sides or are such on the right only (*i.e.*, they are right-hand end points of black intervals). Such points x are represented as

$$x=\frac{c_1}{3}+\frac{c_2}{3^2}+\ldots+\frac{c_n}{3^n}+\ldots$$

where each c is either zero or two, and all the c's from and after some place are not equal to two. Also $f(x)=0$

Let x' be a point within or at the left-hand end of the black interval of the mth stage of subdivision, nearest to x and on the right of it. x' is represented as

$$\frac{c_1}{3}+\frac{c_2}{3^2}+\ldots+\frac{1}{3^m}+\frac{c_{m+1}}{3^{m+1}}+\ldots$$

where the c_r's $(r < m)$ are either zero or two, and c_{m+k} is equal to zero, one, or two (for $k=1, 2, 3\ldots$), and all the c's from and after some place are not equal to two. Also

$$f(x')=\frac{1}{3^{2m}},$$

and
$$(x'-x) > \frac{1}{3^{m+r+1}}$$

where r is finite and equal to the greatest number of 2's that occur one after another in the representation of x.

$$\therefore \quad \lim_{x'\to x} \frac{f(x')-f(x)}{x'-x} \leq \lim_{m\to\infty} \frac{1}{3^{m-r-1}} \leq 0,$$

But, $f(x)=0$, and $f(x')$ is always positive. Therefore the incrementary ratio has the limit zero.

Again if we approach x along a sequence $\{x_m''\}$ of points which are right-hand limiting points of the set.

$$f(x_m'')=f(x)=0,$$

so that
$$\lim_{x''_m\to x} \frac{f(x_m'')-f(x)}{x_m''-x}=0.$$

Therefore at all the points that are right-hand limiting points of the set, the right-hand derivative exists and is equal to zero.

We thus see that at all the points in (0, 1)

$$D^+f(x)=D_+f(x)=0.$$

4

ON VORTEX RINGS IN COMPRESSIBLE FLUIDS

BY

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The only previous author who dealt with vortex rings in compressible fluids is Dr. Chree* whose results however admit of much simplification. As a matter of fact, he attempted to solve the problem of vortex rings of compressible fluids moving in incompressible liquids. The object of the present paper is to discuss the motion of vortex rings of compressible fluids moving in a similar or a different kind of compressible fluids, and I have proved that in the kind of the three dimensional fluid motion in which Stokes's stream function exists, the compressibility of fluids has no affect on the fluid velocity, so that the motion of rings in compressible and incompressible fluids are essentially same. This result, remarkable as it is, was lost sight of by Dr. Chree.

2. Let ρ, ϕ, z be cylindrical co-ordinates of any point referred to the centre of the circular axis of the ring as origin and its axis as z-axis.

ψ=Stokes' stream function, considering motion symmetrical about z-axis,

u, v, w=components of the velocity in cylindrical co-ordinates at the point ρ, ϕ, z, .

Δ=expansion, which is same † in all co-ordinates,

$$=\frac{1}{\rho}\frac{\partial}{\partial\rho}(u\rho)+\frac{1}{\rho}\frac{\partial v}{\partial\phi}+\frac{\partial w}{\partial z},$$

d=density of the fluid,

2ω=vorticity.

It is well known in hydrodynamics that the motion of a fluid which fills infinite space and is at rest at infinity is determinate when Δ and

* Chree—'On vortex rings'—*Proc. Edin. Math. Soc.*, Vol. 6, p. 65-68, 1888.

† Lamb—*Hydrodynamics*, ed. v (1924), p. 48.

ω are known at all points of the region. Thus in Cartesian co-ordinates,* the components of the velocity u, v, w are—

$$u=-\frac{\partial\Theta}{\partial x}+\frac{\partial \mathrm{H}}{\partial y}-\frac{\partial \mathrm{G}}{\partial z}$$

$$v=-\frac{\partial\Theta}{\partial y}+\frac{\partial \mathrm{F}}{\partial z}-\frac{\partial \mathrm{H}}{\partial x}$$

$$w=-\frac{\partial\Theta}{\partial z}+\frac{\partial \mathrm{G}}{\partial x}-\frac{\partial \mathrm{F}}{\partial y}$$

where Θ, F, G, H are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=-\nabla^2\Theta=\Delta=-\frac{1}{d}\frac{\mathrm{D}d}{\mathrm{D}t} \quad \ldots \quad (1)$$

$$\nabla^2\mathrm{F}=-2\xi \quad \ldots \quad (2)$$

$$\nabla^2\mathrm{G}=-2\eta \quad \ldots \quad (3)$$

$$\nabla^2\mathrm{H}=-2\zeta \quad \ldots \quad (4)$$

where ξ, η, ζ are components of ω along the cartesian axes. It is evident from (2), (3), (4) that F, G, H are independent of 'compression' but depend only on the distribution of vorticity while Θ is entirely a function of compression and is independent of vorticity. Hence in finding the motion of the fluid, we can study the effects of vorticity and compression independently.

Now, from the equation of continuity in cylindrical co-ordinates,

$$\frac{1}{d}\frac{\mathrm{D}d}{\mathrm{D}t}+\frac{1}{\rho}\frac{\partial}{\partial\rho}(u\rho)+\frac{\partial w}{\partial z}=0 \qquad (\because \quad v=0)$$

$$\therefore \quad \Delta=-\frac{1}{d}\frac{\mathrm{D}d}{\mathrm{D}t}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(u\rho)+\frac{\partial w}{\partial z}$$

$$=-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\partial\psi}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right) \text{ from (7) } =0$$

$$\left(\because u=-\frac{1}{\rho}\frac{\partial\psi}{\partial z},\ w=\frac{\partial\psi}{\rho\partial\rho}\right)$$

which proves that fluid motion is same as if the liquid is incompressible, in which case the problem has been completely investigated in a previous issue of the *Bulletin of the Calcutta Mathematical Society*. †

* Lamb—*Hydrodynamics*, ed. v, (1924) 208

† See *Bull. Cal. Math. Soc.* Vol. 13, 1922

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On the Expansion of the product of two parabolic cylinder functions in a series of parabolic cylinder funcions

BY

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1. In an issue of the *Proceedings of the Benares Mathematical Society* (Vol. II) Mr. G. Prasad, M.Sc. has obtained an expression for the product of parabolic cylinder functions in a series of parabolic cylinder functions. He has obtained expression of the form

$$D_m(z)D_n(z)$$

$$=D_0(z)\Big[D_{m+n}(z)+m.nD_{m+n-2}(z)+\frac{m(m-1)n(n-1)}{1\cdot 2}D_{m+n-4}(z)$$

$$+\ldots\ldots+\frac{m(m-1)(m-2)2.1.n(n-1)\cdots(n-m+1)}{1.2.3.\ldots m}D_{m-n}z))\Big]$$

and

$$D_m(z)D_n(z)=\sum_{p=0}^{\infty}C_pD_{2p}(z)$$

where

$$C_p=\sum_{r=0}^{m}\frac{m!\ n!}{r!\ (m-r)!\ (n-r)!}a_{n+m-2r,p},$$

$(m+n=\text{even integer})$

and

$$D_m(z)D_n(z)=\sum_{p=0}^{\infty}d_pD_{2p+1}(z)$$

where

$$d_p = \sum_{r=0}^{m} \frac{m!\ n!}{r!\ (m-r)!\ (n-r)!}\ b_{n+m-2r,\ p}$$

$(m+n = \text{odd integer})$,

$a_{n+m-2r,\ p}$ and $b_{n+m-2r,\ p}$ being expressed determinantly.

The first expression is finite while the second expression is infinite. The main difficulty encountered by the author was that assuming an expression $\sum a_r D_r$ for the product of two parabolic cylinder functions, a number of equations involving three consecutive co-efficients were obtained from which it was difficult to obtain the co-efficient of the most general term in the expansion.

The object of the present paper is to give a different finite expression for the product, when the co-efficients in the expansion are deducible from a general law and to evaluate several interesting integrals.

2. We know that $D_n(x)$ satisfies the differential equation,

$$\frac{d^2y}{dx^2} + \left(n+\frac{1}{2} - \frac{1}{4}x^2 \right) y = 0.$$

The recurrence formulæ are

$$D_{n+1}(x) - xD_n(x) + nD_{n-1}(x) = 0$$

and

$$\frac{d}{dx}D_n(x) + \frac{1}{2}\,xD_n(x) - nD_{n-1}(x) = 0.$$

Let

$$y = D_n^2(x).$$

Thus $D_n^2(x)$ satisfies the differential equation

$$\frac{d^3y}{dx^3} + \left(4n+2-x^2 \right)\frac{dy}{dx} - xy = 0 \qquad \ldots \qquad (1)$$

Putting $\xi = \sqrt{2}\,x$, we get the differential equation

$$4\frac{d^3y}{d\xi^3} + \left(8n+4-\xi^2 \right)\frac{dy}{d\xi} - \xi y = 0 \qquad \ldots \qquad (2)$$

To solve this equation, let us assume

$$y=\sum a_r D_r(\xi),$$

the limits of r being determined later on.

Substituting in (2) and making use of the recurrence formulæ, we get after a little simplification,

$$\Sigma a_r[(2r-4n)D_{r+1}+r(4n-2r+2)D_{r-1}]=0$$

If we assume that m is a positive integer, then

$$D_m(x)=e^{-\frac{1}{4}x^2}x^m\left[1-\frac{m(m-1)}{2}\frac{1}{x^2}+\frac{m(m-1)(m-2)(m-3)}{2\cdot 4}\cdot\frac{1}{x^4}-\text{etc}\right]$$

We see that $D_m(x)$ is the product of $e^{-\frac{1}{4}x^2}$ and a terminating series. Hence $D_m{}^2(x)$ is the product of $e^{-\frac{1}{2}x^2}$ and a terminating series. Now $D_r(\xi)$ is the product of $e^{-\frac{1}{2}x^2}$ and a terminating series. This suggests that the upper limit of r is $2n$ and the lower limit zero. The infinite series obtained by putting $r=0$ or $2n+1$ are to be rejected.

Equating to zero, the co-efficients of $D_r(\xi)$, $D_{r-2}(\xi)$, $D_{r-4}(\xi)$ and so on, we get the following relations between the successive coefficients.

$$a_{r-2}(2r-4n-4)+a_r r(4n-2r+2)=0,$$

$$a_{r-4}(2r-4n-8)+a_{r-2}(r-2)(4n-2r+6)=0,$$

$$a_{r-2p}(2r-4n-4p)+a_{r-2p+2}(r-2p+2)(4n-2r+4p-2)=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_0(-4n)+a_2.2(4n-2)=0.$$

Putting $r=2n$, we find that

$$a_{2n-2}=1.n.\quad a_{2n},$$

$$a_{2n-4}=1.3.\frac{n(n-1)}{\underline{|2}}a_{2n},$$

$$a_{2n-6}=1.3.5.\ \frac{n(n-1)(n-2)}{\underline{|3}}\ a_{2n},$$

$$a_{2n-2p}=\frac{1.3.5\ldots(2p-1)n(n-1)\ldots(n-p+1)}{\underline{|p}}\ a_{2n}.$$

Therefore we get,

$$D_n^2(x)=a_{2n}\Big\{D_{2n}(\sqrt{2}x)+1.nD_{2n-2}(\sqrt{2}x)$$

$$+\frac{1.3.n(n-1)}{\lfloor 2}D_{2n-4}(\sqrt{2}x)+\ldots$$

$$+\frac{1.3.5\ldots(2p-1).n(n-1)\ldots(n-p+1)}{\lfloor p}D_{2n-2p}(\sqrt{2}x)$$

$$+\ldots+\frac{1.3.5\ldots(2n-1)n(n-1)(n-2)\ldots1}{\lfloor n}D_0(\sqrt{2}x)\Big\}\quad\ldots\quad(3)$$

Also comparing the co-efficients of x^{2n} from both sides, we find that

$$a_{2n}=\frac{1}{2^n}.$$

3. The above method is not of great advantage in obtaining the expansion of the product $D_m(x)D_n(x)$, when m and n are unequal integers, because we cannot get any relation between two consecutive co-efficients.

We know that

$$D_n(x)^*=(-1)^\mu\ (2\pi)^{-\frac{1}{2}}2^{n+1}\int_{-\infty}^{\infty}e^{\frac{1}{4}x^2-2u^2}\ u^n\begin{matrix}\cos\\ \sin\end{matrix}(2xu)\ du$$

where μ is $\frac{n}{2}$ or $\frac{n-1}{2}$ which ever is an integer, and the cosine or the sine is taken, according as n is an even or an odd integer.

Suppose m and n are both even integers, and $m > n$. Then

$$D_m(x)D_n(x)=(-1)^{\frac{m+n}{2}}\ 2^{m+n+2}(2\pi)^{-1}e^{\frac{1}{2}x^2}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u^m v^n e^{-2(u^2+v^2)}\cos 2xu\cos 2xv\ dudv.$$

* Whittaker and Watson, *Modern Analysis*, p. 348.

Let $u+v=2\phi,\ u-v=2\psi.$

Since $\dfrac{\partial(u,v)}{\partial(\phi,\psi)}=2,$

$$D_m(x)D_n(x)$$

$$=(-1)^{\frac{m+n}{2}}\,2^{m+n+2}\,(2\pi)^{-1}e^{\frac{1}{2}x^2}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\phi+\psi)^m(\phi-\psi)^n e^{-4(\phi^2+\psi^2)}\,\{\cos 4x\phi+\cos 4x\psi\}\,d\phi d\psi$$

If A_{m+n-r} be the co-efficient of $x^{m+n-r}y^r$ in the expansion of $(x+y)^m(x-y)^n$, it can be proved easily that

$$A_{m+n-r}=(-1)^r\sum_{p=0}^{p=r}(-1)^p 2^p\ {}^mC_p\ {}^{m+n-p}C_{r-p}$$

and $$A_{m+n-r}=(-1)^n A_r,$$

where A_r is the co-efficient of $x^r y^{m+n-r}$.

Consequently, we can write

$$D_m(x)D_n(x)$$

$$=(-1)^{\frac{m+n}{2}}\,2^{m+n+2}(2\pi)^{-1}e^{\frac{1}{2}x^2}\sum_{r=0}^{m+n}A_{m+n-r}$$

$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi^{m+n-r}\psi^r e^{-4(\phi^2+\psi^2)}\,\{\cos 4x\phi+\cos 4x\psi\}\,d\phi d\psi$$

Integrating, we find that

$$D_m(x)D_n(x)=\sum\frac{A_{m+n-r}}{2^{\frac{m+n}{2}+1}}\Big\{(-1)^{\frac{r}{2}}\,1.3.5\ldots(r-1)$$

$$\times D_{m+n-r}(\sqrt{2}x)+(-1)^{\frac{m+n-r}{2}}\,1.3.5\ldots(m+n-r-1)D_r(\sqrt{2}x)\Big\}$$

where all the even integral values of r only from zero to $m+n$ are taken.

We can therefore write

$$D_m(x)D_n(x)=\frac{1}{2^{\frac{m+n}{2}}}\Big[A_{m+n}D_{m+n}(\sqrt{2}x)-1.A_{m+n-2}D_{m+n-2}(\sqrt{2}x)$$

$$+1.3.A_{m+n-4}D_{m+n-4}(\sqrt{2}x)+\ldots$$

$$+(-1)^{\frac{m+n}{2}}1.3.5\ldots(m+n-1)A_0D_0(\sqrt{2}x)\Big] \quad \ldots \quad (a)$$

It may be remarked that the same expansion holds good when m and n are both odd integers.

Next let us suppose that m is an odd integer, while n is an even integer, and $m>n$. We have

$$D_m(x)D_n(x)=(-1)^{\frac{m+n-1}{2}}2^{m+n-2}(2\pi)^{-1}e^{\frac{1}{2}x^2}$$

$$\times\sum_{r=0}^{m+n}A_{m+n-r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi^{m+n-r}\psi^{r}e^{-4(\phi^2+\psi^2)}$$

$$\times\{\sin 4x\phi+\sin 4x\psi\}d\phi d\psi$$

After some easy reductions, it is found that

$$D_m(x)D_n(x)$$

$$=\frac{1}{2^{\frac{m+n}{2}}}\Big\{A_{m+n}D_{m+n}(\sqrt{2}x)-1.\ A_{m+n-2}D_{m+n-2}(\sqrt{2}x)$$

$$+1.3.\ A_{m+n-4}\ D_{m+n-4}(\sqrt{2}x)+\ldots$$

$$+(-1)^{\frac{m+n-1}{2}}A_1 1.3.5\cdots(m+n-2)D_1(\sqrt{2}x)\Big\} \quad (b)$$

The same expansion holds good when m is an even integer and n an odd integer.

If in (a) we put $m=n$, and simplify the results, we are led to the experession (3).

(4). From (a), we easily find that

$$\int_{-\infty}^{\infty} D_m(x)D_n(x)D_{m+n}(\sqrt{2}x)\,dx$$

$$= \frac{(\pi)^{\frac{1}{2}}\underline{|m+n}\,A_{m+n}}{2^{\frac{m+n}{2}}}$$

$$\int_{-\infty}^{\infty} D_m(x)D_n(x)D_{m+n-2}(\sqrt{2}x)\,dx$$

$$= \frac{-1.\ A_{m+n-2}\,(\pi)^{\frac{1}{2}}(m+n-2)\,!}{2^{\frac{m+n}{2}}}$$

and so on.

Milne [1] has proved that

$$D_{2n}(2k) = \frac{(-1)^n}{2\sqrt{\pi}}\int_{-\infty}^{\infty} D_{2n}(x)\cos kx\,dx$$

and

$$D_{2n+1}(2k) = \frac{(-1)^n}{2\sqrt{\pi}}\int_{-\infty}^{\infty} D_{2n+1}(x)\sin kx\,dx.$$

Therefore

$$\int_{-\infty}^{\infty} D_{2m}(2k)D_{2n}(2k+2a)\,dk$$

$$= \frac{(-1)^n}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_{2n}(x)D_{2m}(2k)\{\cos kx\cos ax - \sin kx\sin ax\}$$

$$\times\,dx\,dk$$

$$= \frac{(-1)^n}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_{2n}(x)D_{2m}(2k)\cos kx\cos ax\,dx\,dk$$

[1] *Proc. Edin. Math. Soc.* xxxii (1914). pp. 2-14.; xxxiii (1915) pp. 48-64.

$$=\tfrac{1}{2}(-1)^{n+m}\int_{-\infty}^{\infty} D_{2n}(x)D_{2m}(x)\cos axdx$$

$$=\tfrac{1}{2}(-1)^{m+n}\int_{-\infty}^{\infty} \frac{1}{2^{m+n}}\Big[A_{2m+2n}D_{2m+2n}(\sqrt{2}x)$$

$$-1.A_{2m+2n-2}D_{2m+2n-2}(\sqrt{2}x)+1.3.\ A_{2m+2n-4}D_{2m+2n-4}(\sqrt{\ })x$$

$$+\ldots+(-1)^{m+n}1.3.5\ldots(2m+2n-1)A_0D_0(\sqrt{2}x)\Big]\cos ax\,dx$$

$$=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\Big[\frac{1}{2^{m+n}}\Big\{A_{2m+2n}D_{2m+2n}(\sqrt{2}a)$$

$$+1.\ A_{2m+2n-2}D_{2m+2n-2}(\sqrt{2}a)$$

$$+1.3.\ A_{2m+2n-4}D_{2m+2n-4}(\sqrt{2}a)+\ldots$$

$$+1.3.5\ldots(2m+2n-1)A_0D_0(\sqrt{2}a)\Big\}\Big]$$

Similarly

$$\int_{-\infty}^{\infty} D_{2m}(2k)D_{2n+1}(2k+2a)dk$$

$$=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\Big[\frac{1}{2^{m+n+\frac{1}{2}}}\Big\{A_{2m+2n+1}D_{2m+2n+1}(\sqrt{2}a)$$

$$+1.\ A_{2m+2n-1}D_{2m+2n-1}(\sqrt{2}a)$$

$$+1.3.A_{2m+2n-3}D_{2m+2n-3}(\sqrt{2}a)$$

$$+\ldots+1.3.5\ldots(2m+2n-2)A_1D_1(\sqrt{2}a)\Big\}\Big].$$

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General theorem of co-intimacy of symmetrics of a hyperbolic triad

BY

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1. Introduction.

By a hyperbolic triad is meant a group of three elements (points or lines) lying upon a hyperbolic plane. The group may consist either of three lines, two lines and a point, two points and a line or of three points and the elements forming the group may be situated in any manner whatsoever on the plane. The scope of the present paper is to extend to all hyperbolic triads the well-known concurrency theorems of the angle bisectors and the right bisectors of the sides of a triangle formed by three line elements meeting at three actual vertices.

The extension of the angle bisector theorem to all possible triads of linear elements, meeting at actual, improper or ideal vertices was effected by pure Geometry in a paper by S. Mukhopadhyaya and Bhar published in the *Bulletin of the Calcutta Mathematical Society* [Vol. XII. No. 1. 1920-21]. They were first to introduce the concept of the symmetric between two directed lines, and to show that in certain cases it may be a point. The concurrency theorems of the angle bisectors of an ordinary triangle were then shown to be merely particular cases of the general theorem of concurrency of symmetrics between three directed lines.

In the present paper the concept of symmetrics between a point and a line has been first introduced. By the introduction of this important concept which is claimed to be novel, the difficult problem of generalising the concurrency theorem of the right bisectors of the sides of a triangle so as to cover the cases when two or more of the sides do not meet at actual points has been completely solved. Again by the introduction of the concept of *intimacy* it has been possible to entirely abolish the ultra-geometrical concepts of improper, and ideal points, and at the

same time to give to our theorems a larger scope for generality. Some other new terms and concepts have also been introduced which will be found in their proper places. The final result obtained is an elegant geometrical theorem of a highly general character which is applicable to all hyperbolic triads, and further combines in one the two distinct theorems of concurrency, already mentioned. The theorem as well as the numerous deductions which have been made from it, will, it is hoped, prove interesting to all lovers of Non-Euclidean Geometry.

2. LEMMA I:—*If P_1P_2N be perpendicular to O_1NO_2 and if P_1N and P_2N be complementary lengths, then a horocycle through P_1 and P_2 will touch O_1NO_2 at some point O_1 or O_2, such that O_1N or O_2N is complementary to the length $\frac{1}{2}(NP_1+NP_2)$*

Let OX be a tangent to a horocycle at O and P_1N be perpendicular from any point P_1 on the horocycle on OX(Fig. i). Draw OY perpendicular to OX towards the same side on which P_1 lies, so that OY is an axis of the horocycle, drawn from O in the direction of parallelism. Draw P_1L an axis of the horocycle at P_1.

$$\text{Let } ON=x,\quad P_1N=y,\quad OP_1=2z,\quad \angle LP_1O=\angle YOP_1=\phi$$

Then evidently ϕ is the angle of parallelism for the distance Z.

$$\because \quad \tanh z=\cos\phi$$

$$=\sin PON$$

$$=\frac{\sinh y}{\sinh 2z}$$

$$\text{or} \quad \sinh y=2\sinh^2 z$$

$$\text{Again} \quad \cosh 2z=\cosh x\cosh y$$

$$\because \quad 1+\sinh y=\cosh x\cosh y \qquad \ldots \quad \text{(i)}$$

$$\text{or} \quad \cosh x=\operatorname{sech} y+\tanh y \qquad \ldots \quad \text{(ii)}$$

Squaring (*i*) and simplifying, we get

$$\sinh^2 y-2\operatorname{cosech}^2 x\sinh y+1=0. \qquad \ldots \quad (iii)$$

If y_1 and y_2 be two values of y corresponding to a given value of x, we have from (*iii*)

$$\sinh y_1 \sinh y_2=1.$$

Thus if NP_1 cuts the horocycle again at P_2, NP_1 and NP_2 are complementary.

From M the mid-point of P_1P_2 draw MD parallel to P_1L. Then MD is parallel to OY. It follows that the length ON is complementary to the length MN which is $\frac{1}{2}(NP_1+NP_2)$.

Corollary:—If N, P_1, M, P_2 be points taken in order upon a straight line such that M is the mid-point of P_1P_2 and lengths NP_1 and NP_2 are complementary, then cosh MP_1 = sinh MN.

From equation (*i*), $$\cosh x = \frac{1+\sinh y_1}{\cosh y_1} = \frac{1+\sinh y_2}{\cosh y_2};$$

$$\therefore \quad \sinh(y_1-y_2)=\cosh y_1-\cosh y_2,$$

or $$\cosh\tfrac{1}{2}(y_1-y_2)=\sinh\tfrac{1}{2}(y_1+y_2).$$

But $$\tfrac{1}{2}(y_1-y_2)=MP_1 \text{ and } \tfrac{1}{2}(y_1+y_2)=MN$$

Hence $$\cosh MP_1=\sinh MN.$$

3. Definitions. *The principal line of a pair of elements consisting of a point and a line.* Let P a point and AB a line not passing through P. Draw PL perpendicular to AB meeting AB at L. Take P′ on PL such that P′L is complementary to PL, P and P′ lying on the same side of L. Let M be the mid-point of PP′. Then the line perpendicular to PP′ at M is defined to be the *principal* line of P and AB.

It follows from the corollary to Lemma I that cosh MP = sinh ML.

The principal point of a pair of elements consisting of a point and a line. Let P be a point and AB a line not passing through it. Draw PL perpendicular to AB meeting AB at L. Take L′ on PL such that PL′ is complementary to PL, L and L′ lying on the same side of P. Let S be the mid-point of LL′. Then the point S is defined as the *principal point* of P and AB.

It follows from the corollary to Lemma I that cosh SL = sinh SP.

The middle parallel of two parallel lines. The locus of points equidistant from two given parallel lines is a line parallel to both. This line is defined to be the *middle parallel* of the two given lines.

4. LEMMA II:—*If Q be any point on the principal line of P and AB then cosh PQ = sinh QD where D is the foot of the perpendicular drawn from Q on AB.*

Let P be a point and AB any line not passing through P. Let PL be drawn perpendicular to AB, L lying on AB. Let MQ be the

principal line of P and AB, M lying on PL. Let D be the foot of the perpendicular drawn from Q on AB. Join QP (fig. *ii*).

Then
$$\sinh QD = \sinh ML \cosh MQ,$$
$$= \cosh PM \cosh MQ,$$
$$= \cosh PQ.$$

LEMMA III:—*If a line is perpendicular to the principal line of P and AB and if P is the length of the perpendicular from P on this line, then*

$$\sinh p = \cos \phi,\ 1,\ \text{or},\ \cosh d$$

according as the line intersects AB at an angle ϕ, is parallel to it, or possesses a common perpendicular of length d with it.

Let AB, P, L and M be as in Lemma II. Let QC be any line perpendicular to MQ the principal line of P and AB. Let PK be the perpendicular from P on QC such that $PK = p$. Let QC cut AB at an angle ϕ (fig. *iii*), be parallel to it (fig. *iv*), or possess a common perpendicular of length d with AB (fig. *v*)

Then
$$\cos \phi,\ 1,\ \text{or},\ \cosh d = \sinh ML \sinh MQ,$$
$$= \cosh PM \sinh MQ,$$
$$= \sinh p.$$

LEMMA IV:—*If a line passes through the principal point of P and AB, and if p be the length of the perpendicular from P on it, then*

$$\sinh p = \cos \phi,\ 1,\ \text{or},\ \cosh d$$

according as the line intersects AB at an angle ϕ, is parallel to it, or possesses a common perpendicular of length d with it.

Let AB, P and L be as before. Let SH be any line through S the principal point of P and AB, intersecting AB at an angle ϕ (fig. *vi*), parallel to it (fig. *vii*), or possessing a common perpendicular of length d with it (fig. *viii*). Let PR be the perpendicular from P on SH such that $PR = p$.

Then
$$\cosh d,\ 1,\ \text{or},\ \cos \phi = \cosh SL \sin LSH,$$
$$= \sinh SP \sin LSH,$$
$$= \sinh p.$$

5. LEMMA V :—*If PM be a perpendicular to AB from a point P lying in AB and if p be the length of the perpendicular from P on any line parallel to PM (in the sense PM) then*

$$\sinh p = \cos \phi,\ 1,\ \text{or},\ \cosh d$$

according as the line intersects AB at an angle ϕ, is parallel to it, or possesses a common perpendicular of length d with it.

Let P be a point in the line AB. Let PM be drawn perpendicular to AB. Let PQ be a line parallel to PM (in the sense PM). Let PQ intersect AB at an angle ϕ (fig. *ix*), be parallel to it (fig. *x*), or possess a common perpendicular of length d with it (fig. *xi*). Let PL be perpendicular to QP such that PL$=p$.

Then
$$\begin{aligned}\cos \phi,\ 1,\ \text{or},\ \cosh d &= \cosh \mathrm{PL} \sin \mathrm{LPA},\\ &= \cosh \mathrm{PL} \cos \mathrm{MPL},\\ &= \cosh \mathrm{PL} \tanh \mathrm{PL},\\ &= \sinh p.\end{aligned}$$

6. INTIMACY AND CO-INTIMACY.

The elements we will deal with are the point, the line, and the horocycle, the last being representative of a conceptual point to which its axes converge. All horocycles with the same system of axes are equivalent.

A point and a line will be called *intimate* if the former lies on the latter. Two straight lines will be called *intimate* if one is perpendicular to the other. A straight line and a horocycle will be called *intimate* if the line is an axis of the horocycle. A horocycle may be regarded as *intimate* with itself and consequently with any equivalent horocycle.

The *join* of two elements is a third element intimate with both. Between any two distinct elements a unique join always exists. The join of two points is a straight line passing through both. The join of a point and a line is the straight line through the point, perpendicular to the given line (the point may lie on the line). The join of two intersecting straight lines is their point of intersection. The join of two parallel lines is any horocycle of which both the lines are axes. The join of two non-intersecting and non-parallel lines is the line perpendicular to both. The join of a point and a horocycle is that axis of the horocycle which passes through the point.

The join of two horocycles is the straight line which meets both horocycles at right angles and consequently is an axis to either horocycle. The join of a horocycle and a line, not its axis, is that axis of the horocycle which is perpendicular to the given line. The join of a horocycle and a line which is its axis may be taken to be the horocycle itself or any equivalent horocycle.

Any three elements will be called *co-intimate* if there is a common element which is intimate with each. Thus three straight lines passing through the same point are co-intimate as each of them is intimate with this point. Also three straight lines perpendicular to the same straight line are co-intimate. Again three straight lines parallel in the same sense are co-intimate as each of them is intimate with a common horocyle. Two lines and a point are co-intimate if a straight line through the point perpendicular to one of the lines is also perpendicular to the other line. Two points and a line are co-intimate if the straight line passing through the points is perpendicular to the line. Three points are co-intimate if they lie on the same straight line. Two lines and a horocycle are co-intimate if an axis of the horocycle is perpendicular to both the lines. Again two lines and a horocycle are co-intimate if both the lines are axes of the horocycle, for in this case each of the three given elements is intimate with the horocycle itself. A point, a line and a horocycle are co-intimate if the perpendicular from the point to the line, is an axis of the horocyle. Two points and a horocycle are co-intimate if the straight line through the points is an axis of the horocycle. Two horocycles and a line are co-intimate if the common axis of the two horocycles is perpendicular to the line. Two horocycles and a point are co-intimate if the common axis of the two horocycles passes through the point.

It would hardly be appropriate to call the three elements concurrent in all the above cases. We have therefore ventured to introduce the name co-intimacy to cover all these cases and hope that it will be acceptable to Mathematicians. S. Mukhopadhyaya has used already the expression '*range of intimacy* (*co-intimacy*) *of two curves*' for the set of points of intersection of the two curves (*Sir Asutosh Mookerjee Silver Jubilee* Vol. II, 1922).

7. Directed elements.

A point element or a line element may be taken in two opposite senses. With each point P we may associate a clockwise or a counter-clockwise direction of rotation about the point. With each line AB we may associate either the direction AB or the direction BA. We attach

no sense to a horocyclic element. A point or a line taken with a particular direction associated to it we will call a *directed element.*

The sense of a directed line AB *relative* to a point P is defined to be clockwise or counter-clockwise according as the circuit PABP is clockwise or counter-clockwise.

Two directed points having the same sense are called *similarly directed.* They are called *oppositely directed* if they have opposite senses.

A directed point and a directed line are called *similarly directed* if the sense of the line relative to the point is the same as the sense of the point. If these senses are opposite, the point and the line are said to be *oppositely directed.*

Two directed lines parallel to one another are called *similarly directed* if the sense of each is the same as the sense of parallelism or opposite to it. They are said to be *oppositely directed* if the sense of one is the same as the sense of parallelism while the sense of the other is opposite to it.

Two directed lines with a common perpendicular are called *similarly directed* if they have the same sense relative to a point on this common perpendicular produced, while they are said to be *oppositely directed* if their senses relative to such a point are opposite.

8. THE MEASURE OF DIVERGENCE BETWEEN TWO DIRECTED ELEMENTS.

The *divergence* between two directed points at a distance d apart is measured by $-\cosh d$ or $+\cosh d$ according as the points are similarly or oppositely directed.

The *divergence* between a directed point and a directed line at a distance d from it is measured by $-\sinh d$ or $+\sinh d$ according as the point and the line are similarly or oppositely directed. If they are intimate the measure of divergence between them vanishes.

The *divergence* between two directed lines meeting at a point and making an angle δ with one another is measured by $\cos \delta$.

The *divergence* between two directed lines parallel to one another is measured by $+1$ or -1 according as they are similarly or oppositely directed.

The *divergence* between two directed lines with a common perpendicular of length d is measured by $+\cosh d$ or $-\cosh d$ according as the lines are similarly or oppositely directed.

If ρ be a directed element such that the measure of divergence between ρ and a given directed element a is the same as the measure of

divergence between ρ and another directed element β then ρ is defined to be *equidivergent* with α and β. It is evident that if a directed element ρ is equidivergent with the directed elements α and β, as also with the directed elements α and γ, then ρ is equidivergent with β and γ.

9. Horocycles equidivergent with two directed elements.

A horocycle is said to be *equidivergent* with two directed points α and β if an equivalent horocycle passing through α also passes through β and conversely, and if every directed point taken on this equivalent horocycle is either similarly directed to both α and β or is oppositely directed to both.

A horocycle is said to be *equidivergent* with a directed point α and a directed line β if an equivalent horocycle passing through α touches β and conversely, and if every directed point taken on this equivalent horocycle is either similarly directed to both α and β or is oppositely directed to both.

A horocycle is said to be *equidivergent* with two directed lines α and β if an equivalent horocycle touching α also touches β and conversely, and if every directed point taken on this equivalent horocycle is either similarly directed to both α and β or is oppositely directed to both. Again a horocycle is said to be *equidivergent* with two directed lines α and β, if both α and β are axes of the horocycle and are similarly directed.

We shall now show that if H is a horocycle equidivergent with the directed elements α and β and also with the directed elements β and γ then H is equidivergent with β and γ.

In the first case suppose that α and β are not similarly directed parallel lines. Draw a horocycle H′ equivalent to H and passing through α if it is a point or touching α if it is line. Since H is equidivergent with α and β, H′ passes through β if it is a point or touches β, if it is a line. Since H is equidivergent with α and γ, H′ passes through γ, if it is a point or touches γ, if it is a line. Again if P is any point on H′ similarly directed to α, it follows that P is similarly directed to β as well as to γ, and if Q is any point on H′ oppositely directed to α it follows that Q is oppositely directed to β as well as to γ. Hence from definition H is equidivergent with β and γ.

Next suppose α and β to be similarly directed parallel lines. The horocycle touching both α and β is not in this case equidivergent with α and β, as any directed point on this horocycle is oppositely directed to β when it is similarly directed to α. Hence H must be a horocycle

which has both α and β as axes. Since H is equidivergent with α and γ, the latter must be an axis of H and similarly directed to α. It follows that β and γ are both axes of H and are similarly directed. Hence from definition H is equidivergent with β and γ.

10. THE SYMMETRIC BETWEEN TWO DIRECTED ELEMENTS.

Between any two directed elements α and β there exists a unique element $\{\alpha\beta\}$ intimate with all elements (directed points, directed lines, or horocycles) equidivergent with α and β. $\{\alpha\beta\}$ is defined to be the *symmetric* between α and β.

(*i*) *The symmetric between two similarly directed points P and Q is the right bisector of PQ.* Let p be the right bisector. Obviously any directed line or directed point intimate with p is equidistant from P and Q, and is either similarly directed to both P and Q or is oppositely directed to both. It is thus equidivergent with P and Q. Again if there be a horocycle H intimate with p (*i.e.*, having p as an axis) then a horocycle through P having the same system of axes as H, passes through Q. Since P and Q are similarly directed, every directed point on this second horocycle is either similarly directed with respect to both P and Q or is oppositely directed to both. H is then by definition equidivergent with P and Q.

(*ii*) *The symmetric between two oppositely directed points P and Q is the mid-point of PQ.* Any directed line AB intimate with the mid-point is equidistant from P and Q. Since P and Q lie on opposite sidss of AB the circuits PABP and QABQ have opposite senses. Hence if the sense of P is the same as the sense of the circuit PABP the sense of Q is the same as that of QABQ, while if the sense of P is opposite to the sense of PABP the sense of Q is opposite to that of QABQ. In every case therefore AB is similarly directed to both P and Q or oppositely directed to both. It follows that AB is equidivergent P and Q.

(*iii*) *The symmetric between a directed point P and a line AB similarly directed to it is the principal line of P and AB.* Let p be the principal line. Let Q be any directed point on P. Then Q must be on the same side of AB as P since p cannot intersect AB as it possesses a common perpendicular with AB. Hence Q is either similarly directed to both P and AB or is oppositely directed to both. Lemma II then shows that Q is equidivergent with P and AB. Similarly it follows from Lemma III that any line intimate with p (*i.e.*, perpendicular to p) is equidivergent with P and AB. Again Lemma I shows that a horocycle through P having p as an axis touches AB. It is also obvious that every directed point on this horocycle is either similarly directed

to both P and AB or oppositely directed to both, since all points on the horocycle lie on the same side of AB as P. It follows from that any horocycle intimate with p is equidivergent with P and AB.

(*iv*) *The symmetric between a directed point P, and a line AB oppositely directed to it is the principal point of P and AB.* Let S be the principal point. Taking into consideration the directions of the elements concerned, it follows at once from Lemma IV that any directed line intimate with S is equidivergent with P and AB.

(*v*) *The symmetric between a directed point P and a directed line AB intimate with it is a horocycle having as an axis the directed line PL, the sense of AB relative to L being the same as the sense of the directed point P.* Let H be this horocycle. It follows from Lemma V that any directed line intimate with H is equidivergent with P and AB. Again a horocycle H′ equivalent to H and passing through P touches AB at P, All points on H′ lie on the same side of AB as L and therefore the sense of AB relative to every point on H′ is the same as the sense of P. It follows that every directed point on H is either similarly directed to both P and AB or oppositely directed to both Hence all horocycles intimate with H and therefore equivalent to it, are equidivergent with P and AB.

(*vi*) *The symmetric between the directed lines OA and OB meeting at the point O is the external bisector of angle AOB.*

(*vii*) *The symmetric between two similarly directed parallel lines is a horocycle having both the lines as axes.*

(*vii*) *The symmetric between two oppositely directed parallel lines is their middle parallel.*

(*ix*) *The symmetric between two similarly directed lines with a common perpendicular is the mid-point of this perpendicular.*

(*x*) *The symmetic between to oppositely directed lines with a common perpendicular is the line besecting this perpendicular at right angles.*

Conversely it can be shown in every case that a directed point, a directed line or a horocycle equidivergent with the directed elements α and β is intimate with the symmetric $\{\alpha\beta\}$.

11. THEOREM:—*If α, β, γ be any three distinct directed elements (points or lines) on a hyperbolic plane, the symmetrics $\{\beta\gamma\}$, $\{\gamma\alpha\}$ and $\{\alpha\beta\}$ are co-intimate.*

In the first case let $\{\gamma\alpha\}$ and $\{\alpha\beta\}$ have a point or a line element as their join. Call this element ρ and associate a particular direction with it. ρ is then a directed element intimate with the symmetric

between the directed elements γ and α. ρ is therefore equidivergent with γ and α, Similarly ρ is equidivergent with α and β. It follows from Art. 8 that ρ is equidivergent with β and γ. Hence ρ must be intimate with the symmetric $\{\beta\gamma\}$. The symmetrics $\{\alpha\beta\}$, $\{\beta\gamma\}$, $\{\gamma\alpha\}$ are therefore co-intimate each being intimate with the common element ρ.

Next suppose that the join of $\{\gamma\alpha\}$. and $\{\alpha\beta\}$ is a horocyclic element H. H is then equidivergent with γ and α being intimate with the symmetric between them Similarly H is equidivergent with α snd β. It follows from Art. 9 that H is equidivergent with β and γ. It must therefore be intimate with the symmetric $\{\beta\gamma\}$. Hence the symmetrics $\{\alpha\beta\}$, $\{\beta\gamma\}$, $\{\gamma\alpha\}$ are co-intimate each being intimate with the common element H.

12. Summary of Cases

The following are the more important cases of the general theorem proved.

Case I. If a triad consists of three points A, B, C, then

(*a*) The right bisectors of the lines BC, CA and AB either meet at a point, are all parallel in the same sense, or are all perpendicular to a common line.

(*b*) The right bisector of BC meets at right angles the line joining the mid-points of CA and AB.

Case II. If a triad consists of a straight line l and two points B and C lying on the same side of it, then

(*a*) The principal line of B and l, the principal line of c and l, and the right bisector of BC either meet at a point, are all parallel in the same sense, or are all perpendicular to a common line.

(*b*) The principal line of B and l meets at right angles, the line joining the mid-point of BC with the principal point of c and l.

(*c*) The right bisector of BC meets at right angles, the line joining the principal point of c and l.

Case III. If a triad consists of a straight line l and two points B and C lying on opposite sides of it, then

(*a*) The principal point of B and l, the principal point of C and l, and the mid-point of BC lie on the same straight line.

(*b*) The principal line of B and l, and the principal line of C and l possess a common perpendicular passing through the mid-point of BC.

(*c*) The right bisector of BC and the principal line of C and l possess a common perpendicular passing through the principal point of B and l.

Case IV. If a triad consisis of two points B and C and a line l passing through one of the points (say B), and if BL is drawn perdendicular to l, L lying on the same side of l as C, then

(*a*) The right bisector of BC and the princial line of C and l are either both parallel to BL or possess a common perpendicular parallel to BL.

(*b*) The line joining the mid-point of BC with the principal point of C and l is parallel to BL.

(*c*) The perpendicular from the principal point of C and l to the right bisector of BC is parallel to LB.

(*d*) The perpendicular from the mid-point of BC to the principal line of C and l is parallel to LB.

Case V. If a triad consists of a point A and two lines m and n with a common perpendicular PQ P lying on m and Q lying on n), and if A lies between m and n then,

(*a*) The right bisector of PQ, the principal line of A and m and the principal line of A and n, either meet at a point, are all parallel in the same sense or are all perpendicular to a common line.

(*b*) The right bisector of PQ meets at right angles the line joining the principal point of A and m with the principal point of A and n.

(*c*) The principal line of A and n meets at right angles the line joining the mid-point of PQ with the principal point of A and m.

Case VI. If a triad consists of a point A and two lines m and n with a common perpendicular PQ and if the line m lies between A and n then,

(*a*) The principal point of A and m, the principal point of A and n and the mid-point of PQ lie on the same straight line.

(*b*) The principal line of A and m, and the principal line of A and n possess a common perpendicular passing through the mid-point of PQ.

(c) The right bisector of PQ and the principal line of A and m possess a common perpendicular passing through the principal point of A and n.

Case VII. If a triad consists of two lines m and n with a common perpendicular PQ, and a point A lying on m and if AL be drawn perpendicular to m, L lying on the same side of m as n, then

(a) The right bisector of PQ and the principal line of A and n are either both parallel to AL or possess a common perpendicular parallel to AL.

(b) The line joining the mid-point of PQ with the principal point of A and n is parallel to AL.

(c) The perpendicular from the mid-point of PQ on the principal line of A and n is parallel to LA.

(d) The perpendicular from the principal point of A and n to the right bisector of PQ is perpendicular to LA.

Case VIII. If a traid consists of two parallel lines m and n and a point A lying between them, then

(a) The principal line of A and m, the principal line of A and n, and the middle parallel of m and n are either parallel in the same sense, meet at a common point or are all perpendicular to a common line.

(b) The middle parallel of m and n meets at right angles the line joining the principal point of A and m with the principal point of A and n.

(c) The perpendicular from the principal point of A and n to the principal line of A and n is parallel to m and n in the same sense in which they are parallel to each other.

Case IX. If a traid consists of two parallel lines m and n, and a point A such that m lies between A and n, then

(a) The principal line of A and m, and the principal line of A and n possess a common perpendicular, parallel to m and n in the same sense in which they are parallel to each other.

(b) The line joining the principal point of A and m with the principal point of A and n is parallel to m and n in the same sense in which they are a parallel to each other.

(c) The principal line of A and m, and the middle parallel of m and n possess a common perpendicular passing through the principal point of A and n.

Case X. If a traid consists of two parallel lines m and n and a point A lying on m, and if AL be drawn perpendicular to m, L lying on the same side of m as n, then

(*a*) The principal point of A and n lies on a line parallel to AL and to n (in the sense in which n is parallel to m).

(*b*) The principal line of A and n meets at right angles the line parallel to LA and to n (in the sense in which n is parallel to m).

(*c*) The middle parallel of m and n, and the principal line of A and n possess a common perpendicular parallel to AL.

(*d*) The perpendicular from the principal point of A and n to the middle parallel of m and n is parallel to LA.

Case XI. If a traid consists of two lines OA and OB, meeting at the point O, and another point C lying in the angle AOB, then

(*a*) The internal bisector of $\angle$ AOB, the principal line of C and OA and the principal line of C and OB, either meet at a point, are all parallel in the same sense, or are all perpendicular to a common line.

(*b*) The internal bisector of $\angle$ AOB meets at right angles the line joining the principal point of C and OA with the principal point of C and OB.

(*c*) The external bisector of $\angle$ AOB, and the principal line of C and OA, possess a common perpendicular passing through the principal point of C and OB.

Case XII. If a traid consists of two lines OA and OB meeting at a point O, and another point C lying on OA, and if CL be drawn perpendicular to OA, L lying on the same side of OA as B, then

(*a*) The internal bisector of $\angle$ AOB, and the principal line of C and OB are either both parallel to CL or possess a common perpendicular parallel to CL.

(*b*) The perpendicular from the principal point of C and OB to the external bisector of the $\angle$ AOB is parallel to CL.

(*c*) The external bisector of $\angle$ AOB and the principal line of C and OB are either both parallel to LC or possess a common perpendicular parallel to LC.

(*d*) The perpendicular from the principal point of C and OB to the internal bisector of $\angle$ AOB is parallel to LC.

Case XIII. If a triad consists of three lines BC, CA and AB meeting at the points A, B, C then

(*a*) The internal bisectors of the angles BAC, CBA and ACB meet a point.

(*b*) The internal bisector of $\angle$BAC and the external bisectors of $\angle$CBA and $\angle$ACB, either meet at a point, are all parallel in the same sense, or are all perpendicular to the same straight line.

Case XIV. If a triad consists of two parallel lines AL and BM, and a line AB meeting the two former lines at A and B, then

(*a*) The internal bisector of $\angle$LAB, the internal bisector of $\angle$MBA and the middle parallel of AL and BM meet at a point.

(*b*) The external bisector of $\angle$LAB, the external bisector of $\angle$MBA and the middle parallel of AL and BM either meet at a point, are all parallel in the same sense or are all perpendicular to the same straight line.

(*c*) The internal bisector of $\angle$LAB and the external bisector of $\angle$MBA possess a common perpendicular parallel to AL.

Case XV. If a triad consists of two lines AL and BM having a common perpendicular PQ, and a line AB meeting the two former lines at A and B, and if L and M lie on the same side of AB, then

(*a*) The internal bisectors of $\angle$LAB and $\angle$MBA, and the right bisector of PQ either meet a point, are all parallel in the same sense or are all perpendicular to a common line.

(*b*) The internal bisector of $\angle$LAB and the external bisector of $\angle$MBA possess a common perpendicular passing through the mid-point of PQ.

Case XVI. If a triad consists of two lines OP and OQ meeting at O and another line LM such that PL is a common perpendicular to OP and LM and QM is a common perpendicular to OQ and LM, then

(*a*) The internal bisector of $\angle$POQ, the right bisector of PL, and the right bisector of QM meet at a point.

(*b*) The internal bisector of $\angle$POQ meets at right angles to the line joining the mid-points of PL and QM.

(*c*) The external bisector of $\angle$POQ and the right bisector of PL possess a common perpendicular passing through the mid-point of QM.

Case XVII. If a triad consists of three lines AB, CD and EF such that AC is a common perpendicular to AB, and CD, DE is a common perpendicular to CD and EF and FB is a common perpendicular to AB and EF, and if every two of the lines lie on the same side of the third then.

(*a*) The right bisectors of AC, DE and BF either meet at a point, are all parallel in the same sense, or are all perpendicular to a common line.

(*b*) The right bisector of AC meets at right angles the line joining mid-points of BF and DE.

Case XVIII. If a triad consists of three lines, a, b, c, every two of the lines possessing a common perpendicular, and two of the lines, say b and c, lie on opposite sides of a, then

(*a*) The mid-points of the three common perpendiculars lie on the same straight line.

(*b*) The right bisectors of any two of the common perpendiculars themselves possess a common perpendicular, which passes through the mid-point of the third common perpendicular.

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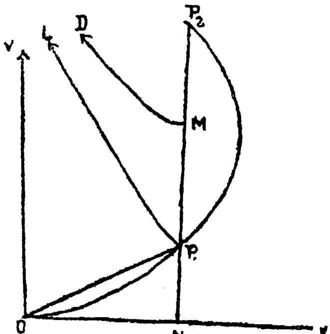
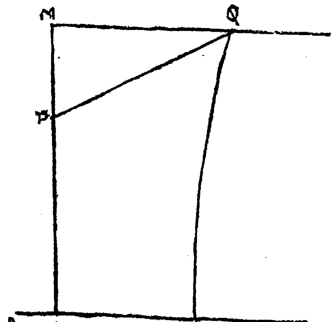


fig i



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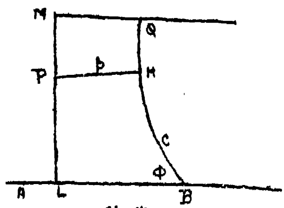

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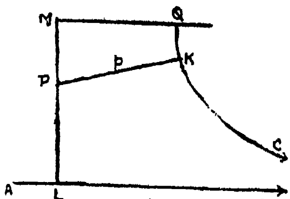

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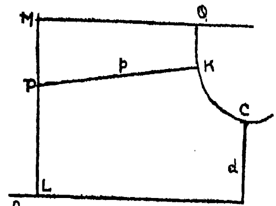

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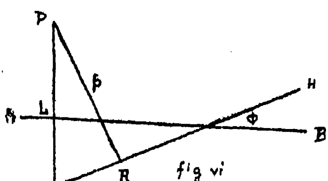

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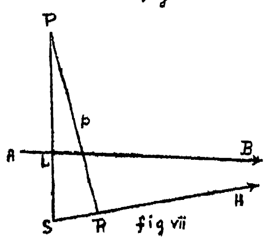
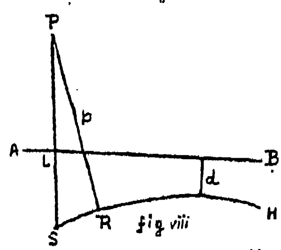


fig vii



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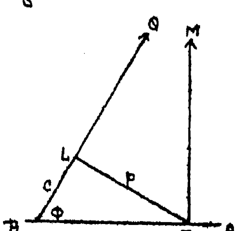

fig ix

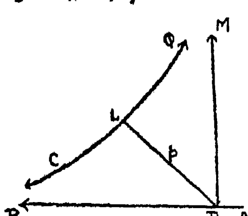

fig x

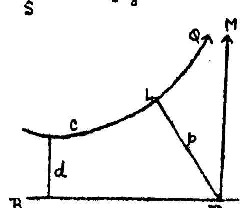

fig xi

CORRIGENDA.

(*Reference.—Bull. Cal. Math. Soc., Vol. XVI, No. 3.*)

Page 133, line	5 (1st para),	*for*	form	*read*	form and otherwise
,, 133, ,,	14 (footnote),	,,	$x^2+z^2-1)$	,,	(x^2+z^2-1)
,, 134, ,,	8,	,,	attained	,,	obtained
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,, 134, ,,	36,	,,	first order	,,	first order, or of a linear partial differential equation of the first order.
,, 135, ,,	22,	,,	$\frac{\partial f}{\partial x}+p\ \frac{\partial f}{\partial z}$	,,	$\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}$
,, 136, ,,	2,	,,	classification is	,,	classification of the integrals of non-linear equation is
,, 136, ,,	10,	,,	non-liner	,,	non-linear
,, 137, ,,	3,	,,	$\{x+ay+\theta(a)\}$	,,	$\{x+ay+\theta(a)\}^2$
,, 137, ,,	4,	,,	$x\{+ay+\theta(a)\}$,,		$\{x+ay+\theta(a)\}$
,, 137. ,,	13,	,,	(3) must become	,,	(3) must become, $3a(\frac{2}{3}y+k+a^2)^{\frac{1}{2}}-y=\theta'(a)$ ... (4)
,, 137, ,,	14,	,,	$3a(\frac{2}{3}y+k+a^2)^{\frac{3}{2}}$	,,	$(\frac{2}{3}y+k+a^2)^{\frac{3}{2}}$
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,, 137, last line,		,,	$9(p^2+q^2)=4$	,,	$9(p^2z+q^2)=4$

1

Two Āryabhaṭas of Al-Biruni

By

Bibhutibhusan Datta

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Introduction

It has been stated by Al-Biruni,* in 1030 A. D., that there were two Indian astronomers bearing the same name Āryabhaṭa. One of them he has distinguished as "the elder Āryabhaṭa" and the other, consequently the younger, as "Āryabhaṭa of Kusumapura." He has warned his readers not to confound between the two and has further added that the latter belonged to the "school" of the former, "he belongs to his followers, for he quotes him and follows his example" (I. 246). No Indian writer, before the celebrated Bhāskarāchārya (*b.* 1114 A. D.), has any reference to two Āryabhaṭas. Even in the case of Bhāskarāchārya, the reference is an indirect one. It will be shown later on that the references of Al-Biruni and of Bhāskarāchārya are not to the same two writers. Hence the statements of Al-Biruni require a very close scrutiny, especially in view of the fact that, on the basis of them, an attempt has been made in recent years to deprive India of much of her originality and antiquity in Astronomy and Mathematics by creating an element of doubt about the date and authorship of a book by one of her earliest known astronomers and mathematicians of note, Āryabhaṭa. The attempt originated with Kaye † and has subsequently been much accentuated by Smith. In his recent book on the history of mathematics, in noticing the works of Āryabhaṭa the elder—he has very little or almost nothing to say of the younger Āryabhaṭa—on no less than five occasions, Smith has warned his readers not to forget the so-called uncertainty about the date and authorship of the *Gaṇita*, a section of his works. ‡ Smith as also others seems to have been influenced by Kaye who would find foreign influence, particularly the

* Al-Biruni's *India*, English trans. by Sachau, Vols. I & II, London (1910).

† Kaye, Āryabhaṭa, *Journ. Asiat. Soc. Beng.* (1908), p. 111.

‡ Smith, *History of Mathematics*, Vol. I (1923), & Vol. II (1925), p. 156; also the footnotes on the pages I. 153, 156; II. 379, 444. *Cf.* also p. 308.

Greek, in all Indian writings.* Al-Biruni's statements, though occasionally emphatic, do not bear the scrutiny of higher criticism and will be found almost confusing in the light of that.

Works of Āryabhaṭa

Al-Biruni has attributed three works to Āryabhaṭa—without specifying which of the two writers of the same name has been meant—*viz.*, *Daśagītikā*, *Āryāṣṭaśata* and *Tantra*. The first two treatises have also been referred to by Brahmagupta† and Al-Biruni must have first learnt of them from his works (*cf.* I. 306). For he has stated that none of these works were available to him and that all his knowledge about Āryabhaṭa was derived "through quotations from him given by Brahmagupta" (I. 370). The other work has not been noticed by any previous Indian writer and we have no further information about it.‡ After Brahmagupta, the works of Āryabhaṭa are known to have been alluded to by Bhaṭṭotpala (966 A. D.), Prithudaka-Swami (*c.* 975 A. D.), Bhāskarāchārya (*b.* 1114 A. D.) and most of his scholiasts such as Ganeśa, Munīśwara and Jñānarāja who lived about the sixteenth century of the Christian era. It appears from the quotations of these writers that Āryabhaṭa's treatises were in their hands. They were, however, lost in subsequent time and in the beginning of the nineteenth century, Colebrooke wrote: "A long and diligent research in various parts of India has, however, failed of recovering any part ... of the

* To gauge the effect of Kaye's propaganda against Indian Mathematics, the following papers may also be consulted: F. Cajori, "The Controversy on the Origin of our Numerals," *Scientific Monthly*, IX. 458: L. N. G. Filon, "The Beginnings of Arithmetic," *Math. Gaz.* (1925), p. 413.

† Brahmagupta, *Brāhma-sphuṭa-siddhānta*, ed. Pandit Sudhakara Dvivedi, Benares (1902), ch. XI, verse 8. Also compare the whole chapter.

‡ Lassen (*Indische Alterthumskunde*, ii. 1136) was of opinion that by *Tantra* Al-Biruni referred to a commentary of the *Sūrya-siddhānta* by Āryabhaṭa (*cf.* Wilson, *Mackenzie Collection*, i. 119) and in this respect he has been further corroborated by Bhāu Dājī (vide *infra*). This surmise is not at all free from doubt. For no Indian writer anterior to Al-Biruni or even to Bhāskarāchārya is known to have alluded to such a book; nor was a copy of it in the hands of Al-Biruni. What is then the source of his information about its existence? Just before his statement about a Tantra by Āryabhaṭa, Al-Biruni has written that there were two classes of literature —*Tantra* and *Karaṇa*—slightly inferior to the *Siddhāntas*. It might be that by *Tantra* he simply meant a work of that class. And we know on the testimony of Brahmagupta and others that the *Āryabhaṭīya* consisting of the *Daśagītikā* and the *Āryāṣṭaśata* is a *Tantra*.

Algebraic and other works of Āryabhaṭa." * The manuscripts of the *Daśagītikā* and the *Āryabhaṭīya* were unearthed for the first time by Bhāu Dājī in 1864.† He detected in them almost all the references and quotations from the works of Āryabhaṭa by later Indian writers. So Bhāu Dāji pronounced his manuscripts to be genuine copies of the original treatises of Āryabhaṭa. According to the accounts given in the opening verse of the *Gaṇita* and in the tenth verse of the *Kālakriyā*, two sections of the *Āryabhaṭīya*, the book was composed by Āryabhaṭa at Kusumapura in the year 499 A. D., at the early age of 23. So he was born in 476 A. D.; and this date has now been generally accepted by scholars. The *Āryabhaṭīya* is the name given by the author to his treatise.

Praise of Āryabhaṭa and his followers

Al-Biruni seems to have been full of admiration for Āryabhaṭa and his followers, who gave him "the impression of really being men of great scientific attainments" (I. 227). He accuses Brahmagupta for undue harshness and animosity against Āryabhaṭa; for he writes: "Now it is evident that that which Brahmagupta relates on his own authority and with which he himself agrees is entirely unfounded; but he is blind to this from sheer hatred of Āryabhaṭa, whom he abuses excessively.......He is rude enough to compare Āryabhaṭa for a worm which eating the wood, by chance describes certain characters in it, without understanding them and without intending to draw them.......In such offensive terms he attacks Āryabhaṭa and maltreats him" (I. 376). These are very harsh words indeed. On another occasion Al-Biruni remarks that what Brahmagupta thinks to be a fault of self-contradiction by Āryabhaṭa is really a fault of the text, not of the author. (I. 168.) Referring to a quotation from Āryabhaṭa by Balabhadra (I. 244-46), Al-Biruni criticises him in strong words and shows that "All that Balabhadra produces is foolish both in words and matters." Again at another place: "To Balabhadra it is just as easy to prefer tradition to eyesight, as it is difficult to us to prefer doubt to a clear proof. The truth is entirely with the followers of Āryabhaṭa......" (I. 227).

In this way Al-Biruni has defended Āryabhaṭa against the calumny and misrepresentation of his unfair critics and that shows his faith in Āryabhaṭa and his high scholarship. On the other hand he accuses

* Colebrooke, *Miscellaneous Essays*, ii. 380.

† Bhāu Dājī, "Brief Notes on the Age and Authenticity of Āryabhaṭa, etc.," *Journ. Roy. Asiat. Soc.* (1864), p. 392.

Brahmagupta who, inspite of being the most distinguished of the Indian astronomers, has, in his opinion, consciously sacrificed the true science by muddling it up with the Paurāṇic legends and popular superstitions, probably to escape those dangers which cost Socrates his life (I. 110 *sqq.*). Owing to all these various reasons, it is very likely, that there grew up in Al-Biruni's mind a lurking suspicion if Brahmagupta had rightly quoted Āryabhaṭa. For he has thrice complained for not having got a copy of Āryabhaṭa's work (I. 370; II. 16, 33) and each time on an occasion of writing about his teachings depending on the quotations by Brahmagupta.

The Book of Āryabhaṭa of Kusumapura

Al-Biruni has several times referred to a book of Āryabhaṭa of Kusumapura * (I. 176, 246, 316, 370) and in such a manner as to indicate that he had a copy of the book in his hand. Sachau thinks that it was a Sanskrit "book by Āryabhaṭa, Junior" (I. Pref., p. xxix; *cf.* p. xl). That is very doubtful. For we miss any such book in the list of Sanskrit works on Astronomy, Astrology, etc. (I. 152-8) which were either seen by Al-Biruni or which came to his knowledge. The list contains only three books by one Āryabhaṭa, *viz.*, the *Daśagītikā*, the *Āryāṣṭaśata* and *Tantra*. It will be noticed that the list is nearly accurate as regards the works of Varāhamihira, Brahmagupta, and of Bhaṭṭotpala. Besides, the list contains the works of several other minor writers. It cannot be said that he meant one of the first two books. We have already heard of his complaints in this respect. Further had it been so, it should have occurred to a critical reader like Al-Biruni to compare Brahmagupta's quotations with his copy. On one occasion (I. 378) he has, however, referred to a small book of Āryabhaṭa of Kusumapura. Sachau failed to decipher the correct reading of the title and took it to be *Al-ntf* (?). This is probably the Arabic rescension of a book the Sanskrit original of which cannot be recognised now (*vide infra*). We do not know if the quotations attributed by Al-Biruni to Āryabhaṭa of Kusumapura have been all culled from this book. There is at least one which, however, seems to have been taken from Balabhadra. That the book was much mutilated in translation and became full of errors, is at once clear from the attempt of Al-Biruni to amend it here and there (*cf.* I. 146).†

* Are we to infer then that the *Āryabhaṭīya* is the book of the elder Āryabhaṭa? *Vide infra,* p. 73.

† About the introduction of Āryabhaṭa's works into Arabia, see Colebrooke, *loc. cit.* pp. 424 sq., 454 sq.

The Ārya-siddhānta—its Date

There is another work called the *Ārya-siddhānta* by an author who calls himself Āryabhaṭa.* This is a larger work of 18 chapters and hence is sometimes called the *Mahā-Ārya-siddhānta* or simply the *Mahāsiddhānta* by contrast with the *Āryabhaṭīya* which is smaller and is then called the *Laghu-Ārya-siddhānta*. In 1863, from an inspection of two imperfect copies of the manuscripts of the *Ārya-siddhānta*, Hall announced that " as reference is made in the *Ārya-siddhānta* to Vṛiddha Āryabhaṭa, there should seem to have been two writers called Aryabhaṭa."† This guess was subsequently confirmed by Kern with the remark that the *Ārya-siddhānta* is a very poor production of a later writer. He even suggested Āryabhaṭa to be only the younger astronomer's " *nom de plume*." ‡ The oriental scholars have differed widely about the age of this writer. According to Bentley,§ he lived in the beginning of the fourteenth century A.D. Bhāu Dāji accepted Bentley. Both of them were wrong. For this writer has been certainly alluded to by Bhāskarāchārya (*b.* 1114 A.D.). || So he must be anterior to the latter. From similarity of the treatment and of forms of certain results in the *Ārya-siddhānta* and Śrīdhara's *Triśatikā*, Pandit Sudhakara Dvivedi infers that the author of the former must be posterior to the latter author.¶ According to Sankar Balakrisna Dikshit, Āryabhaṭā (the younger) lived about 953 A.D.** Sewell puts the date at about 950 A.D.†† We know it that he has not been quoted by the eminent scholiast Bhaṭṭotpala (966 A.D.) who has quoted a host of others. We miss his name in the list of Sanskrit authors and their works prepared by Al-Biruni (I. 152-8). If this omission can be counted as a determining factor as regards the settlement of date, it will have to be said that he is even posterior to Al-Biruni (1030 A.D.).

* *Mahā-siddhānta*, ed. Sudhakara Dvivedi, Benares (1910), ch i, verse 1.

† Hall, " On the Ārya-siddhānta," *Journ. Amer. Orient. Soc.*, vi. 559. *Cf. Mahā-siddhānta*, ch. xiii, verse 14.

‡ Kern, *Bṛhat Saṃhitā*, Calcutta (1865), Pref., p. 59.

§ Bentley, *Historical View of the Hindu Astronomy*, London (1825), p. 128.

|| *Cf. Siddhānta-Śiromaṇi*, Grahagaṇita Spaṣṭādhikāra, verse 65 (Vāsanā-bhāṣya).

¶ *Mahā-siddhānta*, Contents, pp. 22-23. According to Dikshita, Śrīdhara lived before Mahāvīra (853 A.D.), while according to Dvivedi he lived in 991 A.D.

** Sankar Balkrisna Dikshit, *History of Indian Astronomy*, Poona (1896), p. 231.

†† Sewell, *The Siddhāntas and the Indian Calendar*, Calcutta (1924), Pref., p. ix.

Relation between different Āryabhaṭas

It is not an easy task to determine the relation between the two Āryabhaṭas of Al-Biruni, Āryabhaṭa of the *Āryabhaṭīya*, Āryabhaṭa of the *Ārya-siddhānta* and Vṛddha Āryabhaṭa. No previous writer seems to have given necessary and sufficient attention to the solution of this which appears to be a riddle. There have been several partial, probably successful, attempts but none has pushed the matter through to the finish. What we know from the express authority of Al-Biruni is that the *Āryabhaṭīya* is the work of one of his Āryabhaṭas. Bhāu Dāji,* Reinaud,† Kern,‡ Weber § and Winternitz || have all expressly identified it to be the work of Al-Biruni's Āryabhaṭa of Kusumapura who in their opinion was born in 476 A.D. This is, of course, in perfect accord with the accounts given in the book. But all of them are silent about his elder Āryabhaṭa. The only inference from their statements will be that he is an earlier writer. Kaye differed from them. He would separate the *Gaṇita* section of the *Āryabhaṭīya* from the rest and would suppose it to be the work of Al-Biruni's Āryabhaṭa of Kusumapura who, in his opinion, lived in the tenth century A.D. The remaining sections have been attributed by him to the elder Āryabhaṭa of Al-Biruni whom he has accepted as being born in 476 A.D.¶ Kaye has not given any good reason for his speciality in the departure from the unanimously accepted opinion of the previous scholars.

Conjecture of Kaye—Unreliable

The above supposition of Kaye will appear in a sense to be in keeping with the accounts of Al-Biruni as also of the *Gaṇita*. But on closer examination it will be found altogether untenable. First of all it will make the confusion worse confounded by the introduction of fresh complexities. Āryabhaṭa's work has been called the *Āryāṣṭaśata*

* *Loc. cit.*

† Reinaud, *Mémoire sur l'Inde*, Paris (1849).

‡ *Loc. cit.*

§ Weber, *History of Indian Literature*, trans. English by Mann and Zachariae, London (1878), pp. 257-58.

|| Winternitz, *Geschichte der Indischen Litteratur*, Bd. iii (1922), p. 562.

¶ Kaye, "The Two Āryabhaṭas," *Bibl. Math.* xiii. My knowledge of this paper extends up to the abstract which appeared in the *Jahrbuck ü. d. Fortschrite d. Math.* (1909).

by writers posterior to him,—not by himself,—because it consists of 108 verses in the *Āryā* metre. All the manuscripts so far recovered from different parts of India have been found to contain 108 verses including the *Gaṇita*. This is very significant. Kaye's supposition will leave the *Āryāṣṭaśata* incomplete and meaningless. Hence it will be, in fact, creating a fresh difficulty in an attempt to meet one. On the other hand there is the open hint of Brahmagupta that the *Gaṇita* is a section of a work of Āryabhaṭa whom he has criticised.* There are many other cogent arguments against Kaye's conjecture. An exhaustive treatment of the subject of the date and authorship of the *Gaṇita* is, however, not possible here, so will be published separately.† We shall here draw attention to two more facts only in refutation of Kaye's supposition. First, very little is gained by the supposition of Kaye. For, none of the quotations by Al-Biruni from his Āryabhaṭa of Kusumapura is from the *Gaṇita*, whereas nearly all of them have parallels in the other sections of the extant *Āryabhaṭīya* as will be shewn later on. To explain this, new suppositions will have to be made. Secondly, on the testimony of Al-Biruni, the date of his Āryabhaṭa of Kusumapura is not the tenth c^ntury A.D. as has been presumed by Kaye.

Probable date of Āryabhaṭa of Kusumapura—Testimony of Al-Biruni

In criticising a quotation by Balabhadra from Āryabhaṭa, Al-Biruni has remarked: "I do not know which of these two namesakes is meant by Bālabhadra" (I. 246). There is another quotation from "the book of Āryabhaṭa of Kusumapura" which Al-Biruni seems to have taken from a commentary by Balabhadra.‡ From these two accounts we learn that Āryabhaṭa of Kusumapura was anterior to Balabhadra. Unfortunately the date of this latter writer has not been even approximately fixed. He has been several times quoted by Bhaṭṭotpala (966 A.D.), so he must be anterior to him. Further it has been stated by Al-Biruni that a work of Balabhadra (I. 246) as also a work of Āryabhaṭa of Kusumapura was translated into Arabic. Now it is known that "the Arabs learned from India, first under Mansur (A.D. 753-774), chiefly Astronomy and secondly under Harun (786-808), by the especial

* *Brāhma-sphuṭa-siddhānta*, ch. xi, verse 43.

† See the author's forthcoming paper, "*Āryabhaṭa, the author of the Gaṇita.*"

‡ *Cf.* Sachau, II. 327. Sachau seems to suggest that the quotation is from the Arabic translation of Balabhadra's commentary on the *Khaṇḍa-Khādyaka* of Brahmagupta.

influence of the ministerial family Barmak, who till 803 ruled Muslim world, specially medicine and astrology." * Soon afterwards "Arabic literature turned off into other channels. There is no more mention of the presence of Hindu scholars at Bagdad nor of translations of the Sanskrit."† The works of Balabhadra and Āryabhaṭa of Kusumapura were thus probably translated during the first period. Hence both of them must at least belong to the seventh century A.D. An allowance of fifty years to reach the reputation of a writer to a foreign country is not too much. Brahmagupta also lived in the same century (628 A.D.). As Balabhadra wrote a commentary of his *Khaṇḍa-Khādyaka* (665 A.D.) and *Brāhma-sphuṭa-siddhānta* he must be assigned to the latter quarter of that century. Āryabhaṭa of Kusumapura then must either be a contemporary of Balabhadra or anterior to Brahmagupta. It can hardly be accepted that a commentator of note like Balabhadra would commit such sad mistakes, as pointed out by Al-Biruni who had a copy of Balabhadra's book, in referring to a contemporary author. They must have been widely separated in time. So I am inclined to take Āryabhaṭa of Kusumapura to be even anterior to Brhamagupta. The subsequent discussion will also show this latter alternative to be more probable. In any case we get it for certain on the testimony of Al-Biruni that his Āryabhaṭa of Kusumapura must have lived before the eighth century. Hence the speculation of Kaye becomes untenable almost on every account.

The Ārya-siddhānta not a work by any Āryabhaṭa of Al-Biruni

In the midst of the aforesaid speculations and suppositions, it is very refreshing to stumble upon a surer fact, a clear and unambiguous decision. It is this: the author of the *Ārya-siddhānta* is not identical with the author of the *Āryabhaṭīya*; he is a person distinct from the either Āryabhaṭa of Al-Biruni. It is important to note this here lest any one takes it to suggest that as one Varāhamihira is the author of a *Laghu-Jātaka* and a *Bṛhajjātaka*, so one Āryabhaṭa might be the author of a *Laghu-Ārya-siddhānta* as well as a *Mahā-Ārya-siddhānta*. To prove our proposition, the following chief reasons are adduced:—(1) None of Al-Biruni's quotations, with the exception of the one about the four hypothetical equatorial towns and which again is common with almost all the Indian astronomers, has any parallel in the

* Sachau, *loc. cit*, II. 313.

† *Ibid*, Pref., p. xxxii.

Ārya-siddhānta; on the other hand there are few which are contradictory to it. (2) There is nothing in the *Ārya-siddhānta*, nor anything has been discovered elsewhere, to suggest that its author had at any time anything to do with Kusumapura. (3) The author of the *Ārya-siddhānta* has followed the orthodox *Smritis* * whereas the author of the *Āryabhaṭīya* has differed from them in several important matters and this is the main cause of Brahmagupta's fulminations against him (*cf. ii.* 111). Some even suggest that the former has attempted to amend the defects of the latter as pointed out by Brahmagupta.† (4) The two authors have stated different and sometimes even contradictory results in many cases, such as, the arithmetical notation,‡ the value of π; the rotation and dimensions of the Earth, the division of the *kalpa* into *yugas*; the number of revolution of the planets in a *caturyuga*; the number of years in the different *yugas*; the volume of a triangular pyramid and of a sphere; etc. (5) In some cases, though the two authors have given the same correct results, the forms given are different, *e.g.*, the area of a triangle and a circle, the sum of a Progression and connected results; etc. (6) Another very important difference is that the author of the *Ārya-siddhānta* has mentioned of the precession of equinoxes whereas it has been overlooked by the author of the *Āryabhaṭīya* and other writers who followed him up to the 8th century. The number of illustrations can be multiplied at pleasure but jointly with what have been stated before about the probable date of the different authors, those enumerated above are thought sufficient to prove our contention.§

* *Cf. Mahā-siddhānta*, Ch. ii, verse 1.

† *Cf.* Sankar Balkrisna Dikshit, *loc. cit.*, p. 231.

‡ It was a mistake of Kern to say that the author of the *Ārya-siddhānta* has employed "the great Āryabhaṭa's peculiar system of arithmetical notation." It is true that both had had recourse to alphabets for arithmetical notation; but their systems are altogether different.

§ *Cf.* Jogeshchandra Roy, *Āmāder Jyotiṣī o Jyotiṣa*, Calcutta (1903), pp. 281-188. Roy remarks that the author of the *Ārya-siddhānta* has followed the left-to-right mode of putting down the numerical notations whereas the author of *Āryabhaṭīya* has kept to the right-to-left mode. His second statement is not strictly true, for on closer examination it will be found that the latter writer has followed different modes in different instances and in some he has mixed up the two modes. In fact, it is not essential to stick to any one mode in his notation. This is so far as the alphabetical numeral notation is concerned. In certain sections of his book, the younger Āryabhaṭa has employed word symbol numerals where the usual right-to-left mode has been adopted.

Vṛddha Āryabhaṭa

Almost all the writers are unanimous in identifying Vṛddha Āryabhaṭa of the *Ārya-siddhānta* with the author of the *Āryabhaṭīya*, Al-Biruni's Aryabhaṭa of Kusumapura. But Sudhakara Dvivedi thinks him to be a different person.* Though he has not pushed the matter further towards the identification of the exact person,—it is not an easy task, indeed—there seems to be at least some truth in his conjecture. The author of the *Ārya-siddhānta* says :†

"Thus for the good of others, the celestial mechanics is spoken in my own words; part equal to the ancient science (āgama) is spoken. *Vipras* should read it, none else.

"What from the *Siddhānta* profounded by Vṛddha Āryabhaṭa had been distorted in recensions, through long ages, I have specified in my own words."

In short, on his own accounts, his work is a revised edition of the work of Vṛddha Āryabhaṭa with necessary emendation of variant readings, parts rewritten and parts retained in original. Hence there should be expected more than a general resemblance between the writings of the two authors. But it has been just pointed out that there are fundamental differences, of principles as well as of results, between the two books; no part, not a line, is common. It can be even suspected if the two authors do really belong to the same school. It is stated by the author of the *Ārya-siddhānta* that his teachings are alike the teachings of Parāśara and the two *Siddhāntas*, his and Parāśara's, were written when a little of the *Kaliyuga* was passed.‡ By his *Siddhānta*, he must have meant the *Siddhānta* which he was editing. Now it is believed that the *Parāśara-siddhānta* was recast into its present form before the beginning of the Christian Era.§ Hence Vṛddha Āryabhaṭa lived, in all probability, near about that age. Too much compactness of the *Āryabhaṭīya*, sometimes even to become ambiguous and unintelligible,|| has led some writers to suspect that its author had before him another larger and detailed work from which he had made an abridged handbook.¶ Is it the work of Vṛddha

* *Cf. Mahā-siddhānta*, Contents, p. 22.

† *Mahā-siddhānta*, Ch. xiii, verses 13, 14.

‡ *Ibid*, Ch. ii, verses 1, 2.

§ Jogeschandra Roy, *loc. cit.*, p. 55. The original *Siddhānta* is believed to have been composed many centuries earlier.

|| Brahmagupta has drawn attention to it on several occasions; *vide Brāhma-sphuṭa-siddhānta*, ii. 19, 33, 47, v. 21, 25; *Khaṇḍa-Khādyaka*, ii. 10.

¶ *Cf.* Kaye, "Aryabhaṭa," p. 116; *Pañcha-siddhāntikā*, Pref. p. vi sq.

Āryabhaṭa? There are certain other facts in support of this conjecture. It has already been stated that Brahmagupta had strongly denounced Āryabhaṭa in the *Brāhma-sphuṭa-siddhānta*. This book was written in his early days at the age of 30, in 628 A.D. In his later years (665 A.D.) he wrote a Karaṇa, the *Khaṇḍa-Khādyaka*, with the object of giving results equal to those of Āryabhaṭa and of bringing them up-to-time.* It is strange to find a confirmed and unrelenting critic to follow the footprints of his erstwhile adversary in later years. It has been suggested that the change came under the pressure of public opinion. A very poor compliment to the genius of Brahmagupta, the most distinguished of Indian mathematicians and astronomers, before Bhāskarāchārya! And still more so when we know it for certain that there was no persecution in India for convictions, spiritual or temporal. Moreover, curiously enough "the dimensions of the epicycles and the positions of the apogees assumed in the Khaṇḍa-Khādyaka (as well as in the sixteenth chapter of the Pañca-siddhāntikā) differ, all of them, more or less from those recorded in the Laghu-Āryabhaṭīya." † All these incongruities will clarify on the supposition that Brahmagupta followed Vṛddha Āryabhaṭa who is not the same as the author of the *Āryabhaṭīya*. It should be remarked that we do not find anything in the *Khaṇḍa-Khādyaka* which may be put against this supposition. Vṛddha Āryabhaṭa should be a follower of orthodox *Smritis* like his follower, the author of the *Ārya-siddhānta*. We cannot say if he was the elder Āryabhaṭa of Al-Biruni. It is more likely that the author of the *Āryasiddhānta* alluded to the author of the *Āryabhaṭīya* as having destroyed the teachings of Vṛddha Āryabhaṭa in his recensions by going against the *Smritis* and introducing other innovations. We do not wish to proceed further with the discussion in this place. We only remark that in this way we shall get three mathematicians and astronomers bearing the same name Āryabhaṭa instead of two as heretofore believed.

Analysis of Al-Biruni's quotations

We shall next proceed to scrutinise all references by Al-Biruni to Āryabhaṭa, whether the elder or the younger, either by a simple mention of the name or by quotation of a passage meant to represent his teaching, and see what can be gathered out of them. There are

* *Khaṇḍa-Khādyaka*, ed. Pandit Babua Misra, Calcutta University (1925), Ch. i, verses 1, 2.

† *Pañca-siddhāntikā*, ed. Thibaut and Dvivedi, Benares (1889), p. xx., *cf.* the note by Schram, Sachau, II. 367.

altogether 38 such references; of these 15 are simple mentions of the name Āryabhaṭa and the remaining 23 are meant to represent the teachings of Āryabhaṭa.

Firstly, analysing according to the sources, we find that Al-Biruni has drawn from the following sources :—

(1) Quotation by Brahmagupta :—
I. 156, 168², 276, 280, 370, 373, 376, 377, 386² ;
II. 16, 17, 33, 111, 190(?).

(2) Quotation by Balabhadra :—I. 244², 245, 246².

(3) Quotation by Puliśa :—I. 266, 275(?).

(4) Quotation by the Arabic writer, Abu-alhsan of Al'ahwas :—
II. 18, 19.

(5) "Book of Āryabhaṭa of Kusumapura"—I. 176, 246, 316, 330, 335, 370.

(6) Unknown Source :—I. 156, 157, 225, 227, 267, 268².

In the last group are included all those references which could not be rightly put into any of the other groups. It should, however, be remarked that, in it, there are two passages (I. 225, 227) which have been stated to be embodying the opinion of "the followers of Āryabhaṭa," so they are probably from the 5th source. Two references (I. 156, 157) are simply to the name, Āryabhaṭa.

Secondly, analysing according to authorship, we get,

(1) Āryabhaṭa the elder :—I. 246, 370².

(2) Āryabhaṭa of Kusumapura :—I. 176, 246, 316, 330, 335, 370.

(3) Āryabhaṭa :—I. 156², 157, 168², 225, 227, 244, 245, 246², 266, 267, 268², 275, 276, 280, 373, 376, 377, 386²; II. 16, 17, 18, 19, 33, 111.

Thus we find that a great majority of Al-Biruni's references cannot be directly assigned to the one or the other of his two Āryabhaṭas. In the last group, are included three references to "the followers of Āryabhaṭa." It cannot be ascertained who have been meant by the epithet.

Comparison with the Āryabhaṭīya: Discrepancies explained

On comparison with the extant *Āryabhaṭīya*, it is found that almost all the quotations by Al-Biruni have parallels in that book. In some cases whole sentences are identical. Thus we can restore the correct reading of a doubtful word occurring in a quotation from *Al-ntf* (I. 370-71).

Sachau reads it with a query as *durtama*; while the correct reading according to the *Āryabhaṭīya* should be *duḥsama*.* It should be remarked that the quotations of this class are directly from the book of Āryabhaṭa of Kusumapura which Al-Biruni had with him. In the case of others which are taken from secondary sources agreement is not so exact. In a few, a sentence or a part of it agree completely but it is mixed up with other useless or wrong matters for which Al-Biruni has reprimanded his authorities. Sometimes it is the practice with Al-Biruni to put his own explanatory words in the mouth of his authority and to pass them as original quotations (*cf.* I. 76; II. 110-11, 337). In such cases agreement will be more in ideas than in words. In one or two instances of this kind a few wrong words have crept in. Thus in the book of Āryabhaṭa of Kusumapura, the names of the eighth and ninth "places" in the arithmetical numeration have to be called *Koṭi-padma* and *Parapadma* respectively whereas in the *Āryabhaṭīya* they are *Koṭi* and *Arbuda*. It may be noted that there are "places" such as *Padma* and *Mahā-padma*. Again in a list of nine datas prepared by Al-Biruni from Abu-alhasan of Al'ahwas, giving the revolutions of the planets according to Āryabhaṭa—"*in the years of al-arjbhar*, i.e., *catur-yuga*"—what have been recorded as the revolutions of the planets Mercury and Venus should be, in fact, the revolutions of their apses. It may be noted that the translator have been led into this error by his defective knowledge of Sanskrit. For even in the original these have not been expressly stated as such only to save repetition. One quotation has been condemned by Al-Biruni as being mixed up with what are foolish (I. 244).† There is another (I. 316) which must be a fabrication. ‡ These little discrepancies might have been due to the text

* *Āryabhaṭīya*, *Kāla-Kriyā-pāda*, verse 9. *Cf. Gaṇaka Taraṅgiṇī*, p. 7.

† Leaving out the foolish additions, we get a parallel in the *Āryabhaṭīya* for the substratum.

‡ Kern has not suspected the correctness of this quotation and, on the other hand, considers it from a book different from the *Āryabhaṭīya* as it does not contain it. The passage in question is from a book of Āryabhaṭa of Kusumapura and it contains a reference to Prithuswāmi, who is no other than the eminent scholiast of Brahmagupta, Prithudaka-swāmi. He lived about 975 A. D. If this statement of Al-Biruni be correct, Āryabhaṭa of Kusumapura must have been a contemporary of Al-Biruni and in that case his acquaintance with the original should have been more thorough. But it has been shewn elsewhere on the basis of accounts given by Al-Biruni himself, that Āryabhaṭa of Kusumapura lived before the eighth century at the latest. That is why we consider this quotation to be a fabrication. Dikshit suggests that Al-Biruni might have taken it from a commentary of Āryabhaṭa and confounded it for an original passage.

of Al-Biruni, or to the Arabic translators of the original Sanskrit works, or to the copyists. His Indian informants might have been responsible for some of the dilatations and distortions. But such faults are not unusual. Al-Biruni has drawn attention to several such instances some of which he has amended. Referring to Alfazārī and Yakub Ibn Tarik who introduced Indian Astronomy into Arabia (*c.* 772 A. D.), Al-Biruni remarks that they even misunderstood the word Āryabhaṭa (II. 19) and further adds: "If we compare these secondary statements with the primary statements of the Hindus, we discover discrepancies, the causes of which is not known to me," (II. 15; *cf.* I. 246, 308). About Arab manuscript tradition he remarks: "The Arabic copy seems to be corrupt like all other books of this kind which I know. Such corruption must of necessity occur in our Arabic writing more particularly at a period like ours, when people care so little about the correctness of what they copy" (I. 162-3). Making due allowances in view of the remarks made above, it must have to be admitted that agreement is more than what can be ordinarily expected under similar circumstances. We, however, do not find parallel for one quotation in the extant *Āryabhaṭīya* (I. 225).* It will be noticed that the passage is more of the nature of a philosophical speculation than an astronomical fact. However, leaving out the two fabricated pieces, we find parallels in the extant *Āryabhaṭīya* for twenty quotations of Al-Biruni out of a total number of twenty-one. Hence we come to the conclusion that, though Al-Biruni has collected about Āryabhaṭa from various sources and under different designations, the *Āryabhaṭīya* is the prime source of all his informations.

Āryabhaṭa of Kusumapura—a Myth

Our next attempt shall be to ascertain the authorship of the *Āryabhaṭīya* to which, it has been just shewn, all the quotations of Al-Biruni, in the name of either of his Āryabhaṭas, can be traced. The author was certainly, anterior to Brahmagupta who, it has been

* It is a quotation from Brahmagupta referring to "the followers of Āryabhaṭa." I fail to trace it in the *Brāhma-sphuṭa-siddhānta* and the *Khaṇḍa-Khādyaka.* There are other similar quotations attributed to Brahmagupta which are not found in his works. There are some which have been much dilated. We learn from Bhāskarāchārya (*Siddhānta Śiromani,* Madhyamādhikāra, Kakṣādhyāya, Śloka 2) that such were the opinions of the Purāṇas. So it appears that Al-Biruni has thrust the words of the Purāṇas into the mouth of Āryabhaṭa through Brahmagupta. Hence neglecting this, the agreement is complete.

already stated, has mentioned by name the two parts of it, *viz.*, the *Daśagītikā* and the *Āryāṣṭaśata*. And it appears on a closer examination of certain passage (I. 370) as if it is the hint of Al-Biruni that Brahmagupta has quoted "the elder Āryabhaṭa." Who is then his Āryabhaṭa of Kusumapura? Al-Biruni's assertions about his separate existence are emphatic. But facts as given by him do not warrant such an assumption. For all quotations in the name of the latter are found in the book of the former. There is no doubt that there existed in India, before the time of Al-Biruni, at least two authors, by name Āryabhaṭa, one of whom was born in 476 A.D. and the other lived about 950 A.D. But there is absolutely nothing to indicate that Al-Biruni was acquainted with the works of the latter author; for none of his quotations from Āryabhaṭa of Kusumapura has any parallel in the work of this author; on the other hand, there is at least one quotation which is directly in contradiction.* Further it has been shewn on the testimony of Al-Biruni that his Āryabhaṭa of Kusumapura cannot be put to such a late age as the tenth century. Is "Aryabhaṭa of Kusumapura" then a different Āryabhaṭa? Such an inference would not be at all safe. It is also noteworthy that there is no reference to two Āryabhaṭas in the works of Indian writers before the celebrated Bhāskarāchārya. Under the circumstances it appears to me that Al-Biruni's Āryabhaṭa of Kusumapura is a pure myth, the one and the same individual has been stated by confusion as two distinct persons. Sankar Balkrisna Dikshit is almost of the same opinion.†

Objections met

In making an assertion to spirit away the reality of Al-Biruni's second Āryabhaṭa one must meet such objections: (1) the mention of *Al-ntf* (?), a book of Āryabhaṭa of Kusumapura; and (2) positive assertions of Al-Biruni. Before Al-Biruni came to India, he had acquired a good general knowledge of Indian Mathematics and Astronomy through Arabic translations. The faithfulness of these translations have been questioned even by Al-Biruni himself. We have already drawn attention to some very sad mistakes in them. *Al-ntf* seems to be an imperfect translation of a few extracts from the works

* For example compare Al-Biruni's *India* I. 370 and *Mahā-siddhānta*, Ch. i, verse 15.

† *Loc. cit.*, p. 310, footnote.

of Āryabhaṭa who lived at Kusumapura in 476 A.D. Probably there were not many things common to this book and the references in Brahmagupta's works which were also translated. So that Al-Biruni failed to recognise the identity of the two writers. He had, however, recognised sufficient resemblance to call the one a follower of the other. Thus Al-Biruni's first wrong impressions about the existence of two Āryabhaṭas originated while he was in Arabia. In India, too, he failed to correct and improve upon these impressions. For inspite of all his attempts he could not procure a copy of the works of Āryabhaṭa. He wrote: "I have found it very hard to work my way into the subject (astronomy), although I have a great liking for it and although I do not spare either trouble or money in collecting Sanskrit books from places where I supposed they were likely to be found, and in procuring for myself, even from very remote places Hindu scholars who understand them and are able to teach me" (I. 24). Again that he could not even secure a better type of Hindu scholars to teach him, inspite of all his earnest efforts, will be at once evident from his remark: "They are niggardly in communicating that which they know, and they take the greatest possible care to withhold it from men of another caste among their own people, still much more, of course, from any foreigner" (I. 22). With such imperfect knowledge it is not strange for one to get confused and commit sad mistakes. Nor should his positive assertions deter us to arrive at a proper and logical conclusion. In this respect it will be sufficient to draw attention to the fact that he has made many such baseless assertions.* He has put in the mouth of Brahmagupta something which is not found in his works. Above all he has even attributed a book of auguries and prophecy to the Great Buddha (I. 158).

* *Cf.* Sachau, II. 265. 337; Pref., xxvi.

2

On the evaluation of a Factorable continuant of Muir *

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From the well-known centro-symmetric factorable continuant of Sylvester †, Muir ‡ has derived a very interesting factorable continuant having certain well-defined characteristics. There is another factorable continuant, also discovered by Muir, which possesses some of those characteristics, *viz.*, §

* For references on the subject see—

Painvin, "Sur un certain systeme d'equations lineaires." *Liouville Journ*, iii, pp. 41-46.

Muir, "Continuants resolvable into linear factors," *Trans. Edin. Roy. Soc.* 41, (343-358), 1905. "Note on Palmstrom's Generalisation of Lame's Equation." *Trans. S. Afric- Phil. Soc.*, 15 pt. 1, (29-33) 1904. "A continuant resolvable into Rational Factors" *Proc. Edin. Roy Soc.*, 24, (105-112) 1901-1902. "Further Note on continuants. *Ibid* vol. 8, (1873-74).

Metzler, "Some Factorable continuants" *Ibid* 34, 1914 (223-229).

Haripada Datta, "On the Failure of Heilermann's Theorem" *Proc. Edin. Math. Soc.*, Vol. 35, Pt. 2 (1916-17). "On the Theory of continued Fractions" *Ibid* Vol. 34, pt, 2, 1915-16.

For further references see,

"On the Evalution of some Factorable contiuuants" Bull. Cal. Math. Soc., Vol. xiii, pt. 71-84 (1922-28) and Vol. xiv, pp. 91-106 (1922-24).

"On a Foctorable continuant" *Ibid* Vol. xiv, pp. 219-38 (1923-24).

"On Two Pairs of Factorable continuants" *Ibid*, Vol. xv, pp. 127-38 (1924-25).

† Sylvester, "Theorème sur les determinants de M. Sylvester." *Nouv. Ann. de Math*, xiii, p. 305 (1854).

‡ Muir, "Factorizable continuants" *Trans. S. Afric. Phil. Soc.*, 15, pt. 1. 1904 (29-33).

§ ,, "Further Note on the Factorizable continuants." *Ibid* xv., pp. 183-94 (1905).

$$\left|\begin{array}{cccccc}
\alpha, & \frac{(2n-1)\beta}{2n-1}, & & & & \\
\frac{1\cdot(\beta+2n-2)}{3-2n}, & \alpha, & \frac{(2n-2)(\beta+1)}{2n-3} & & & \\
& \frac{2(\beta+2n-3)}{5-2n}, & \alpha, & \frac{(2n-3)(\beta+2)}{2n-5} & & \\
& & \frac{3(\beta+2n-4)}{7-2n}, & \alpha & \ldots & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
& & & \ldots\ \alpha, & \frac{(n+1)(\beta+n-2)}{3} & \\
& & & \frac{(n-1)(\beta+n)}{1}, & \alpha+\frac{n(\beta+n-1)}{1} &
\end{array}\right|_n$$

$$=(\alpha+\beta)(\alpha+\beta+2)(\alpha+\beta+4)\ldots\ldots(\alpha+\beta+2n-2).$$

Muir has simplified this identity into

$$\left|\begin{array}{cccccc}
A, & \frac{1\cdot 2n}{2n-1} & & & & \\
\frac{(2n-1)2}{3-2n}, & A, & \frac{2(2n-1)}{2n-3} & & & \\
& \frac{(2n-2)3}{5-2n}, & A, & \ldots & & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
& & & \ldots\ A, & \frac{(n-1)(n+2)}{3} & \\
& & & \frac{(n+1)n}{-1}, & A+\frac{n(n+1)}{1} &
\end{array}\right|_n$$

$$=(A+2)(A+4)(A+6)\ldots\ldots(A+2n) \qquad \ldots \quad (1$$

But he did not proceed to the final proof on account of the growing difficulties. " Unfortunately the set of column-multipliers necessary for effecting this rosolution is not simple, the first line of it being

$$1,\ 2-\frac{1}{n},\ 3-\frac{3}{n},\ 4-\frac{6}{n},\ \ldots, n-\frac{n(n-1)}{2n}\ ."$$

In this paper is given a complete proof of this last identity. The operations necessary for effecting this proof have led to some fresh algebraic relations, The most general factorable oontinuant which is resolvable by means of the same operations has also been obtained. There is also given another identity which is analogous to a well-known identity of the Calculus of Finite Differences.

1. If the continuant be written in the reverse order, the identity (1) takes the form,

$$\left|\begin{array}{cccccc} A+\frac{n(n+1)}{1}, & \frac{n(n+1)}{-1} & & & & \\ \frac{(n-1)(n+2)}{3}, & A, & \frac{(n-1)(n+2)}{-3} & & & \\ & \frac{(n-2)(n+3)}{5}, & A, & \frac{(n-2)(n+3)}{-5}, & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & \frac{2(2n-1)}{2n-3}, & A, & \frac{2(2n-1)}{3-2n} \\ & & & \frac{1\cdot 2n}{2n-1}, & A & \end{array}\right|_n$$

$$=(A+2)(A+4)(A+6)\ldots\ldots(A+2n).$$

On the left-side of this identity perform n successive operations of the type

$$m_1\ \mathrm{col}_1+m_2\ \mathrm{col}_2+m_3\ \mathrm{col}_3+\ldots+m_n\ \mathrm{col}_n,$$

in which, for the kth operation

$$\left.\begin{aligned} m_r &= \frac{\binom{r+k-2}{-1}_{2k-2}(n-r+1)(n+r)}{\binom{2k-2}{-2}_{k-1}} \\ \text{and} \qquad m_1 &= \frac{(n-k+1)(n+k)}{A+2k-2} \end{aligned}\right\} \quad \ldots \quad (2)$$

where $\left(\begin{matrix} n \\ \pm c \end{matrix}\right)_r = n(n\pm c)(n\pm 2c)\ldots(n\pm \overline{r-1}c),$

$$\left(\begin{matrix} n \\ \pm c \end{matrix}\right)_r = 1 \text{ ; if } r=0, \text{ or negative}$$

and $\left(\begin{matrix} n \\ -1 \end{matrix}\right)_r = 0;$ if $r > n$, a positive integer.

In the first operation m_1 is governed by the rule for m_r but not in subsequent operations. m_r vanishes for all operations from $r+1$th to nth. After the n operations have been performed, it will be fouud that all the elements, except the last of the first column vanish. Hence the value of the continuant readily follows.

(i) In the particular case when $n=6$, the six operations that are obtained from the above formula are :—

$$42 \text{ col}_1 + 40 \text{ col}_2 + 36 \text{ col}_3 + 30 \text{ col}_4$$
$$+ 22 \text{ col}_5 + 12 \text{ col}_6 = \text{col}_1^{(1)}$$

$$\frac{40}{A+2} \text{ col}_1^{(1)} + 40 \text{ col}_2 + 108 \text{ col}_3 + 180 \text{ col}_4$$
$$+ 220 \text{ col}_5 + 180 \text{ col}_6 = \text{col}_1^{(2)}$$

$$\frac{36}{A+4} \text{ col}_1^{(2)} + 0 \text{ col}_2 + 108 \text{ col}_3 + 450 \text{ col}_4$$
$$+ 990 \text{ col}_5 + 1260 \text{ col}_6 = \text{col}_1^{(3)}$$

$$\frac{30}{A+6}\,\text{col}_1^{(3)}+0\,\text{col}_2+0\,\text{col}_3+450\,\text{col}_4$$

$$+2310\,\text{col}_5+5040\,\text{col}_6=\text{col}_1^{(4)}$$

$$\frac{22}{A+8}\,\text{col}_1^{(4)}+0\,\text{col}_2+0\,\text{col}_3+0\,\text{col}_4$$

$$+2310\,\text{col}_5+11340\,\text{col}_6=\text{col}_1^{(5)}$$

$$\frac{12}{A+10}\,\text{col}_1^{(5)}+0\,\text{col}_2+0\,\text{col}_3+0\,\text{col}_4$$

$$+0\,\text{col}_5+11340\,\text{col}_6=\text{col}_1^{(6)}.$$

2. After each operation it will be noticed that a factor will be common to each element of the first column. If this factor be removed and the resulting determinant be lowered in order, then the aforesaid operations are to be stated thus :—

For the kth operation

$$M_r=\frac{m_{r+k-1}}{m_k}\ * \qquad \cdots \quad (3)$$

where M_r is the multiplier of the rth column and m's have the same meaning as given above.

The most general continuant resolvable by means of the operations (3), is obtained in the following manner :—

On the continuant

$$\left|\begin{array}{cccccc} A+\alpha, & \beta_1 & & & & \\ \gamma_1, & A, & \beta_2 & & & \\ & \gamma_2, & A, & \beta_3 & & \\ \cdots & & \cdots & & \cdots & \cdots \\ & & & \gamma_{n-2}, & A, & \beta_{n-1} \\ & & & & \gamma_{n-1}, & A \end{array}\right|_n$$

perform the first of the operations (3) *viz.*

$$\text{col}_1+\frac{(n-1)(n+2)}{n(n+1)}\,\text{col}_2+\ldots+\frac{1\cdot 2n}{n(n+1)}\,\text{col}_n.$$

* The first of the operations (3) is the same as given by Muir.

Then, by reason of the removability of the factor

$$A+a+\frac{(n-1)(n+2)}{n(n+1)}\beta_1,$$

we find a set of $n-1$ equations, *viz.*

$$\frac{(n-k+1)(n+k)}{n(n+1)}\gamma_k+\frac{(n-k-1)(n+k+2)}{n(n+1)}\beta_{k+1}$$

$$-\frac{(n-k)(n+k+1)}{n(n+1)}a=\frac{(n-k)(n+k+1)(n-1)(n+2)}{n^2(n+1)^2}\beta_1 \quad \dots \quad (4)$$

where k is a positive integer varying from 1 to $n-1$.

From the first of the equations (4), we have

$$a=\frac{(n-2)(n+3)}{(n-1)(n+2)}\beta_2-\frac{(n-1)(n+2)}{n(n+1)}\beta_1+\frac{n(n+1)}{(n-1)(n+2)}\gamma_1 \quad \dots \quad (5)$$

Then eliminating a from the other equations of (4) we have a set of $n-2$ equations, *viz.*

$$\frac{(n-k)(n+k+1)}{n(n+1)}\gamma_{k+1}+\frac{(n-k-2)(n+k+3)}{n(n+1)}\beta_{k+2}$$

$$-\frac{(n-k-1)(n+k+2)(n-2)(n+3)}{(n-1)n(n+1)(n+2)}\beta_2$$

$$=\frac{(n-k-1)(n+k+2)}{(n-1)(n+2)}\gamma_1 \quad \dots \quad (6)$$

where k is a positive integer varying from 1 to $n-2$.

The resulting determinant can be lowered in order, being in fact

$$\begin{vmatrix} A-\frac{(n-1)(n+2)}{n(n+1)}\beta_1, & \beta_2 & & & & \\ \gamma_2-\frac{(n-2)(n+3)}{n(n+1)}\beta_1, & A, & \beta_3 & & & \\ -\frac{(n-3)(n+4)}{n(n+1)}\beta_1, & \gamma_3, & A, & \beta_4 & & \\ \dots & \dots & \dots & \dots & \dots & \\ -\frac{2(2n-1)}{n(n+1)}\beta_1, & & & \gamma_{n-2}, & A, & \beta_{n-1} \\ -\frac{1\cdot 2n}{n(n+1)}\beta_1, & & & & \gamma_{n-1}, & A \end{vmatrix}_{n-1}$$

On this perform the second of the operations (3), *viz.*

$$\mathrm{col}_1+\frac{3(n-2)(n+3)}{(n-1)(n+2)}\,\mathrm{col}_2+\dots\dots+\frac{(n-1)n.2n}{2(n-1)(n+2)}\,\mathrm{col}_{n-1}.$$

Then making another factor

$$A-\frac{(n-1)(n+2)}{n(n+1)}\beta_1+\frac{3(n-2)(n+3)}{(n-1)(n+2)}\beta_2$$

removable from the first column, we get another set of $n-2$ equations, *viz.*,

$$\frac{k(k+1)}{2}\cdot\frac{(n-k)(n+k+1)}{(n-1)(n+2)}\gamma_{k+1}$$

$$+\frac{(k+2)(k+3)}{2}\cdot\frac{(n-k-2)(n+k+3)}{(n-1)(n+2)}\beta_{k+2}$$

$$-\frac{3(k+1)(k+2)}{2}\cdot\frac{(n-k-1)(n+k+2)(n-2)(n+3)}{(n-1)^2(n+2)^2}\beta_2$$

$$=-\frac{k(k+3)}{2}\cdot\frac{(n-k-1)(n+k+2)}{n(n+1)}\beta_1 \qquad \dots \quad (7)$$

where k is a positive integer varying from 1 to $n-2$.

From (6) and (7) we get $2n-4$ equations involving $2n-2$ quantities *viz.*, all β's and all γ's. Then in terms of β_1 and γ_1, all other β's and γ's can be easily expressed thus:—

From (6) and (7) when $k=n-2$, we have two equations, from which eliminating γ_{n-1} we have

$$\beta_2 = \frac{n}{2(n+3)}\gamma_1 + \frac{(n-1)(n+2)^2}{2n(n+1)(n+3)}\beta_1 \qquad \dots \quad (8)$$

With the help of (8) eliminating β_2 from (6) and (7) and solving for γ_{k+1} and β_{k+2} we have

$$\left.\begin{aligned} \beta_k &= \frac{3(k-1)n(n-k+1)(n+k)}{2(2k-1)(n-1)(n+2)(n+k+1)}\gamma_1 \\ &\quad + \frac{(k+1)(n+2)(n-k+1)(n+k)}{2(2k-1)n(n+1)(n+k+1)}\beta_1 \\ \gamma_k &= \frac{3(k+1)n(n-k)(n+k+1)}{2(2k+1)(n-1)(n+2)(n-k+1)}\gamma_1 \\ &\quad + \frac{(k-1)(n+2)(n-k)(n+k+1)}{2(2k+1)n(n+1)(n-k+1)}\beta_1 \end{aligned}\right\} \qquad \dots \quad (9)$$

From (5) we have

$$\alpha = \frac{3n^2}{2(n-1)(n+2)}\gamma_1 - \frac{(n+2)}{2(n+1)}\beta_1 \qquad (10)$$

We then get the following general theorem.—

The continuant

$$\begin{vmatrix} A+\alpha, & \beta_1 & & & & \\ \gamma_1, & A, & \beta_2 & & & \\ & \gamma_2, & A, & \beta_3 & & \\ \dots & & \dots & & \dots & \dots \\ & & & \gamma_{n-2}, & A, & \beta_{n-1} \\ & & & & \gamma_{n-1}, & A \end{vmatrix}_n$$

$$=\left\{ A+\frac{3n^2}{2(n-1)(n+2)}\gamma_1+\frac{(n+2)(n-2)}{2n(n+1)}\beta_1 \right\}$$

$$\times\left\{ A+\frac{3n(n-2)}{2(n-1)(n+2)}\gamma_1+\frac{(n+2)(n-4)}{2n(n+1)}\beta_1 \right\}$$

$$\times\left\{ A+\frac{3n(n-4)}{2(n-1)(n+2)}\gamma_1+\frac{(n+2)(n-6)}{2n(n+1)}\beta_1 \right\}$$

$$\ldots\times\left\{ A-\frac{3n(n-2)}{2(n-1)(n+2)}\gamma_1-\frac{(n+2)n}{2n(n+1)}\beta_1 \right\}$$

where a, β's and γ's are given by (9) and (10). The factors are in A. P., the common difference being

$$\frac{3n}{(n-1)(n+2)}\gamma_1+\frac{n+2}{n(n+1)}\beta_1.$$

Cor. If

$$\gamma_1=\frac{(n-1)(n+2)}{3} \quad \text{and} \quad \beta_1=-n(n+1),$$

we come back to the continuant of Art. 1.

3. Denote

$$T_n=\binom{2r}{-1}_n+\sum_{k=1}^{k=n}(-)^k\frac{(r+1)(r-k+1)\binom{r}{-1}^2_{k-1}\binom{2r-k}{-1}_{n-k}}{(k)!},$$

$$D_{a,n}=\sum_{k=1}^{k=\frac{n-1}{2}}\left\{\frac{(a-4k+3)(a-2n)\binom{a-2k-n}{-1}_{n-2k-1}}{(n-2k+2)!}\right\}+1$$

where n is an odd positive integer;

$$D'_{a,n}=\sum_{k=1}^{k=\frac{n}{2}}\left\{\frac{(a-4k+3\binom{a-2k-n}{-1}_{n-2k+1}}{(n-2k+2)!}\right\}+1$$

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where n is an even positive integer, and

$$\phi(r, p)=S_{r,p}+\sum_{k=1}^{k=p}(-)^k\frac{(r+1)(r-k+1)^2\binom{r}{-1}^3_{k-1}}{\binom{2r}{-1}_k(k)!}S_{r-k,\,p-k},$$

where $S_{r,p}$ is the sum of the products of r factors 1, 2, 3...r, taken p at a time. Then we have the identities.

(*i*) $$T_n=(-)^n\frac{r(r-n)\binom{r-1}{-1}^2_{n-1}}{(n)!}$$

Proof. Since

$$T_{n+1}=(2r-n)T_n+(-)^{n+1}\frac{(r+1)(r-n)\binom{r}{-1}^2_n}{(n+1)!},$$

the theorem follows by induction.

Cor. $T_n=0$, if $n\geq r$

(*ii*) $$D_{a,n}=\frac{\binom{a-n}{-1}_{n-1}}{(n)!}$$

Proof:—If the last two terms of the series for $D_{a,n}$ be added together, it will be found that $(a-2n+3)(a-2n+2)$ will be common factors. Remove these and again add the last two terms. On continuing this process successively for $\frac{n-1}{2}$ times, the theorem will follow.

(*iii*) $$D'_{a,n}=\frac{\binom{a-n}{-1}_n}{(n)!}$$

The proof is similar to (*ii*).

(*iv*) $\phi(r, p)=\phi(r-1, p)$

$$+\sum_{k=1}^{k=m}\frac{r^2\binom{r-1}{-1}^3_{2k-1}(r-2k)}{\binom{2r}{-1}_{4k}}\phi(r-2k-1, p-2k)\ \ldots \quad (11)$$

where $m=\frac{p-1}{2}$ or $\frac{p}{2}$ according as p is odd or even.

$$Proof.—S_{r,\,p}=\sum_{k=0}^{k=p}\binom{r}{-1}_k S_{r-k-1,\,p-k} \qquad \dots \quad (12)$$

for $$S_{r,\,p}=S_{r-1,\,p}+r\,S_{r-1,\,p-1}.$$

Substitute values of S's from (12) in the left-side of (11), then we have

$$\phi(r, p)=S_{r-1,\,p}+\sum_{k=1}^{k=p}\frac{\binom{r}{-1}_k}{\binom{2r}{-1}_k}T_k\,S_{r-k-1,\,p-k}.$$

$$=S_{r-1,\,p}+\sum_{k=1}^{k=p}(-)^k\frac{r^2(r-k)\binom{r-1}{-1}^3_{k-1}}{\binom{2r}{-1}_k(k)!}S_{r-k-1,\,p-k} \quad \dots \quad (13)$$

by (i)

Again substituting the values ϕ's, the right-side of (11)

$$=S_{r-1,\,p}-\frac{r(r-1)}{2}S_{r-2,\,p-1}+\frac{r^2(r-1)^3(r-2)}{\binom{2r}{-1}_4}D'_{2r,\,2}\;S_{r-3,\,p-2}$$

$$-\frac{r^2(r-1)^3\binom{r-2}{-1}^2_2}{\binom{2r}{-1}_4(2r-6)}D_{2r,\,3}\quad S_{r-4,\,p-3}+\dots$$

$$=S_{r-1,\,p}+\sum_{k=1}^{k=p}(-)^k\frac{r^2(r-k)\binom{r-1}{-1}^3_{k-1}}{\binom{2r}{-1}_k(k!}S_{r-k-1,\,p-k}$$

by (ii) and (iii).

$\therefore$ the theorem (iv) is established.

Cor. $\phi(r, p)=0$, if $p \geq r$. This follows readily from (12), for;

$$S_{r,\,p}=0, \quad \text{if} \quad p > r.$$

(*v*) If $\phi(r, p)=\phi(r-1, p)$ and $p < r-1$,

then
$$\phi(r, p)=0.$$

Proof:—Since $\phi(p, p)=0$ by (*iv*) Cor, this theorem may be easily proved by adding all the equations that may be obtained by putting $r, r-1, r-2, \ldots (p+2)$ and $(p+1)$ for r in the given condition.

4. $\phi(r, p)=0$, if p is odd and $< r$. ... (14)

Proof:—By actual calculation we find that

$$\phi(r, 1)=0, \quad \text{for} \quad S_{r,\,1}=\frac{r(r+1)}{2}.$$

$$\therefore \quad \phi(r, 3)=\phi(r-1, 3), \quad \text{by } (iv).$$

$$\therefore \quad \phi(r, 3)=0, \quad \text{by } (v),$$

Proceeding in this manner, the general case may be proved.

The case when p is even and $< r$, presents a difficulty in getting the result in a suitable form.

5. $$\sum_{k=0}^{k=r} \frac{\begin{pmatrix} n-r+k-1 \\ -1 \end{pmatrix}_k \begin{pmatrix} n+r+1 \\ -1 \end{pmatrix}_k}{\begin{pmatrix} 2r \\ -1 \end{pmatrix}(k)!} = \frac{\begin{pmatrix} n+r \\ -1 \end{pmatrix}_{2r}}{(2r)!} \quad \ldots \quad (15)$$

Proof:—Put $n=p$, where $p=0, 1, 2, 3 \ldots$, or $r-1$; then right-side$=0$, the left-side

$$=\frac{1}{\begin{pmatrix} 2r \\ -1 \end{pmatrix}_{r-p-1}} \sum_{k=0}^{k=r-1} (-)^k \; {}^{r-p}C_k \begin{pmatrix} 2r-k \\ -1 \end{pmatrix}_{r-p-1} = 0$$

by difference formulae. If $p= -1, -2, -3, \ldots$, or $-r$; right side$=0$; the left side

$$=\frac{1}{\begin{pmatrix} 2r \\ -1 \end{pmatrix}_{r+p}} \sum_{k=0}^{k=r+p+1} (-)^k \begin{pmatrix} 2r-k \\ -1 \end{pmatrix}_{r+p} {}^{r+p+1}C_k=0,$$

by difference formulae.

If $n=r$, either side $=1$.

Thus the equation is satisfied for $2r+1$ values of n. So it is an identity.

6. Let us obtain, by using a as the multiplying factor, the successive orders of differences from any function of x, *viz.*, $\phi(x)$ in the following manner.*

Let $d_k\phi(x)$ be used to denote the kth order of differences, then

$$d_1\phi(x)=\phi(ax)-\phi(x)=\psi(x), \text{ say,}$$

$$d_2\phi(x)=\psi(ax)-a\psi(x)=Q(x) \quad ,,$$

$$d_3\phi(x)=Q(ax)-a^2Q(x)=f(x) \quad ,,$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$d_r\phi(x)=F(ax)-a^{r-1}F(x) \quad \text{where} \quad F(x)=d_{r-1}\phi(x).$$

It can be easily shown that if $\phi(x)$ be rational integral function of x of the nth degree, then

$$d_{n+1}\phi(x)=0. \qquad \ldots \quad (16)$$

7. Denote

$$\begin{bmatrix} a^ny+d \\ \pm c \end{bmatrix}_r$$

$$=(a^ny+d)(a^{n\pm c}y+d)(a^{n\pm 2c}y+d)\ldots(a^{n\pm\overline{r-1}c}y+d);$$

$$\begin{bmatrix} a^ny+d \\ \pm c \end{bmatrix}_0=1,$$

and
$$\begin{bmatrix} a^n-1 \\ -1 \end{bmatrix}_n=(a^n-1)(a^{n-1}-1)(a^{n-2})\ldots(a-1)\dagger$$

Then, if $\phi(x)$ be a rational integral function of x of the nth degree, it can be developed thus

$$\phi(x)=\sum_{k=0}^{k=n}\left|\frac{d_k\phi(x)}{a^{\frac{1}{2}(k-1)k}\,x^k}\right|\frac{\begin{bmatrix} a^{k-1}x-\delta \\ -1 \end{bmatrix}_k}{\dfrac{\delta}{a^k}\begin{bmatrix} a^k-1 \\ -1 \end{bmatrix}_k {}^kS_k}\ddagger \qquad \ldots \quad (17)$$

* See "On the Evaluation of some Factorable continuants, part II." *Bull. Cal. Math. Soc.*, Vol. xiv, pp. 91-106 (1923-24).

† The right-side had previously been denoted by $\begin{bmatrix} n \\ 1 \end{bmatrix}$.

‡ See Boole, *Calculus of Finite Differences*, 2nd Ed., pp. 11-12.

where ${}^{r}S_k$ denotes the sum of the products of r factors $1, a, a^2, \ldots a^{r-1}$ taken k at a time and

$$\left| \frac{d_k \phi(x)}{a^{\frac{1}{2}(k-1)k} x^k} \right|_{\frac{\delta}{a^k}}$$

denotes that the kth order of differences obtained from $\phi(x)$ by using a as the multiplying factor, is to be divided by $a^{\frac{1}{2}(k-1)k} x^k$ and then in the result x is to be made equal to $\frac{\delta}{a^k}$.

Proof :—Assume

$$\phi(x)=A_0+A_1\frac{x-\delta}{(a-1)^1 S_1}+A_2\frac{(x-\delta)(ax-\delta)}{(a^2-1)(a-1)^2 S_2}$$

$$+A_3\frac{(x-\delta)(ax-\delta)(a^2x-\delta)}{(a^3-1)(a^2-1)(a-1)^3 S_3}+\ldots \qquad \ldots \quad (18)$$

then $$d_1\phi(x)=A_1x+A_2\frac{(ax-\delta)x}{(a-1)a}+A_3\frac{(ax-\delta)(a^2x-\delta)x}{(a^2-1)(a-1)a^3}+ \quad \ldots \quad (19)$$

$$d_2\phi(x)=A_2ax^2+A_3\frac{(a^2x-\delta)x^2}{(a-1)a}+\ldots \qquad \ldots \quad (20)$$

and thus obtain other successive orders of differences. Then putting $x=\delta$ in (18), $x=\frac{\delta}{a}$ in (19), etc., we get A_0, A_1, etc.

Cor. $$\begin{bmatrix} a^{r-1}x+1 \\ -1 \end{bmatrix}_r = \sum_{k=0}^{k=r} {}^{r}S_k \begin{bmatrix} a^{-k+1}+\delta \\ +1 \end{bmatrix}_k \begin{bmatrix} a^{r-k-1}x-\delta \\ -1 \end{bmatrix}_{r-k} \S.$$

§ See "On a Factorable continuant," *Bull. Cal. Math. Soc.*, Vol. XIV, pp. 219-38, Art. 1 (1923-24).

3

BHĀSKARĀCHĀRYA AND SIMULTANEOUS INDETERMINATE EQUATIONS OF THE FIRST DEGREE *

BY

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1. From the printed editions of Bhāskarāchārya's Līlāvatī and Bījagaṇita it appears that he considered only one case of simultaneous indeterminate equations of the first degree, *viz.*,

$$\frac{ax \pm c}{b} = y, \qquad \frac{dx \pm e}{b} = z,$$

in which the denominators of the left-hand expressions are the same.

The problem of which the first equation is an algebraic statement would be stated by Bhāskarāchārya as follows:

What integer being multiplied by the integer a and then increased or decreased by another integer c will be divisible by the integer b without remainder?

So a is called a multiplier (*guṇaka*) and b a divisor (*hara* or *hāra*) of the required integer, and c is called a *kṣepa* (*lit.*, what is thrown into, or away from, something, *i.e.*, what is added to or subtracted from something). The *kṣepa* is positive (*dhana*) or negative (*ṛṇa*) according as it is added to, or subtracted from the product of the required integer and its given multiplier. As the product ax, when increased or decreased by c, is to be divisible by b, a is also called the dividend (*bhājya*) in which case the required integer is called its multiplier (*guṇa*) most probably to satisfy the metre of the stanza or stanzas giving the rule.

* One of the three essays to which the Griffith Memorial Prize for 1922 was adjudged by the Calcutta University.

2. The stanza dealing with the above-mentioned case of simultaneous indeterminate equations begins thus:

" एकौ हरश्चेद् गुणकौ विभिन्नौ "

i.e., if the divisors be the same but the multipliers different.

This raises the question: Has not Bhāskarāchārya considered the case in which the divisors are also different? The object of this paper is to attempt an answer to this question.

3. The writer of this paper has found this case dealt with in two palm-leaf manuscripts (both in Oriya characters) of Līlāvatī with the commentary of Śrīdhara Mahāpātra,* an Oriya mathematician, who completed the commentary in the year 1639 of the Śaka era (1717 A. D.).

4. Through the courtesy of Mahāmahopādhyāya Sadāśiva Miśra of Puri the writer was able to have on loan the manuscript copies referred to above. One of the copies may still be found in the library of the *Mukti-maṇḍapa Sabhā* of Puri, of which the Mahāmahopādhyāya is the president.

5. On a request made by the present writer through Professor Jadunath Sarkar, the Oriental Libraries at Mysore, Madras, and Travancore were scrutinised and as a result the general case of simultaneous indeterminate equations of the first degree has been found to occur in two palm-leaf manuscript copies of the text of Līlāvatī, both in Andhra characters—one † in the Oriental Library, the State Library for such manuscripts, Mysore and the other ‡ in the Government

* Śrīdhar Mahāpātra was an inhabitant of Balapur, a village still known by that name and lying to the north of the Puri town. His father was Nimā Mahāpātra and his mother Gauri.

Nimā Mahāpātra also was a mathematician and gave the following rule for the area of a segment of a circle:

व्यासार्द्धेन शरोनेन हतं ज्यार्द्धं विशोधयेत् ।
व्यासपादहताच्चापार्द्धे षं स्याद्धनुषः फलम् ॥

One fails to find any reference to Śrīdhara Mahāpātra or his father in late M. M. Sudhakara Dvidedi's *Gaṇakataraṅginī* (Accounts of Indian astronomers and mathematicians) or in Rai Bahadur Joges Chandra Rai's *Āmāder Jyotiṣī O Jyotiṣa*, although both these works give accounts of Indian mathematicians and astronomers of even more recent times.

† Manuscript No. 1335, pp. 42 and 43.

‡ Folio 28(*b*) of Ms. described under 13480.

Oriental Manuscripts Library) Madras. The same rule and the same example in illustration thereof occur with slight variations in all the manuscripts referred to above.

संश्लिष्टबहुसामान्यकुट्टकसूत्रम् :—

हारे विभिन्ने गुणके च भिन्ने
स्वादाद्यराशेर्गुणकस्तु साध्य: ।
द्वितीयभाज्यघ्नतदाद्यजो गुण: *
क्षेपो भवेत् क्षेपयुतो द्वितीये † ॥
द्वितीयभाज्यघ्नतदाद्यहारो
भाज्यो भवेत्तत्र हरो हर: स्यात् ‡ ।
एवं प्रकल्प्यापि च कुट्टकेऽथ
जातो गुणश्चाद्यहरेण निघ्न: ॥
गुणो भवेदाद्यगुणेन युक्तो §
हरघ्नहारोऽत्र हर: प्रदिष्ट: ।
अथ तृतीयेऽपि तथैव कुर्या-||
देवं बहूनामपि साधयेत्तु ¶ ॥

अत्रोदाहरणमाह ।

क: सप्तनिघ्नो विहृतो द्विषष्ट्या
चिकावशेषोऽथ स एव राशि: ।
षड़ाहत: सैकशतेन ** भक्त:
पञ्चाग्रकश्चाथ †† स एव राशि: ॥
अष्टाहत: सप्तशताङ्कभक्तो ‡‡
नवाग्रको मे वद राशिसंख्याम् ।
धनाग्रकेणापि तदेव राशे:
किं स्याद्धनं कुट्टविधानमाद्य: ॥

* द्वितीयभाज्यघ्नगुणस्स चाद्य: । (Madras and Mysore copies.)

† The Mysore copy has द्वितीय: for द्वितीये ।

‡ भाज्यो भवेत्तत्र हरोऽपरस् स्यात् । (Madras copy.)

§ For युक्तो the Madras copy has युतो and the Mysore copy निघ्नो. The Mysore reading is evidently wrong in this case.

|| The Mysore copy has तदेव for तथैव.

¶ The Madras and Mysore copies have साधयीत for साधयेत्तु.

** चैकशतेन (Madras and Mysore copies).

†† पञ्चाग्रकश्चापि (,, ,, ,, ,,).

‡‡ सप्तदशैर्विभक्तो (,, ,, ,, ,,).

Translation

Rule for the solution of simultaneous indeterminate equations of the first degree :—

When the given divisors and multipliers are different in the various conditions * to be satisfied in a complex problem on *kuṭṭaka* (*i.e.*, when the required integer has to satisfy various conditions leading to indeterminate equations of the first degree), an integer satisfying the first condition should, first of all, be found. The integer thus obtained, being multiplied by the (known) dividend in the second condition and then increased by the *kṣepa* of the second condition, becomes the *kṣepa* of a new (*lit*, second) problem (the first condition being taken as the first problem), the (known) dividend (*i.e.*, multiplier) being the product of the divisor of the first condition and dividend of the second condition, and the divisor being the same as that of the second condition. With these *kṣepa*, dividend, and divisor, a multiplier should be found by the rule of *kuṭṭaka*. The product of this multiplier and the divisor of the first condition, being increased by the (previously unknown) multiplier satisfying the first condition (as originally obtained), is the required integer satisfying the first two conditions. (If many values of the common multiplier be desired or the *kṣepas* be negative or there be three conditions in the problem), proceed similarly taking, as instructed by previous teachers,† the product of the divisors of the first two conditions as the divisor. Proceed similarly if there be more than three conditions in the problem.

Example.

(A) What is the integer,

(1) which, being multiplied by 7 and then divided by 62, leaves 3 as remainder,

(2) which, being multiplied by 6 and then divided by 101, leaves 5 as the remainder, and

(3) which, being multiplied by 8 and then divided by 17, leaves 9 as the remainder ?

(Here the *kṣepas* are negative)

(B) What is the process when the *kṣepas* are positive ?

* Each condition gives rise to an indeterminate equation.

† The word *pradiṣṭa* has been interpreted by by the commentator in this way.

7. *Solution according to the rule.*

First, suppose that the *kṣepas* are positive so that the problem would stand thus:

What is the integer,

(1) which, being multiplied by 7 and then *increased* by 3, becomes divisible by 62,

etc., etc. ?

By the rule of *kuṭṭaka* or otherwise it may be seen that 35 satisfies the first condition of the problem (*i.e.*, as modified above). Multiply 35 by 6 (dividend of the second condition) and add to the product 5 (the *kṣepa* of the second condition). The result is 215. This is taken as the *kṣepa* of a new problem on simple *kuṭṭaka*, the dividend and divisor being respectively $372(=62\times 6)$ and 101 by the rule quoted above. These *kṣepa*, dividend and divisor give 73 as a multiplier by the rule of simple *kuṭṭaka*. Hence, by the above rule, the integer satisfying the first two conditions $=62\times 73+35=4561$.

The original problem consists of two parts, (A) and (B). In the first part the *kṣepas* are all negative. Hence we shall have to use the rule योगजे तक्षणाच्छुद्धे गुणाप्ती स्तो वियोगजे * (In the case of a negative *kṣepa* the multiplier and the result are obtained by subtracting respectively from the divisor and the dividend the multiplier and the result given by a positive *kṣepa*. So an integer which will satisfy the first two conditions of the first part of the problem is obtained by subtracting 4561 from the divisor. But what is this divisor ? The rule says that it is 6262—the product of the divisors 62 and 101 of the first two conditions. Hence an integer which will satisfy the first two conditions of the first part of the problem $=6262-4561=1701$. Other values of the common multiplier may be obtained by adding 4561 or 1701 (according as the *kṣepas* are positive or negative) to the product of the divisor 6262 (by the rule) and any assumed integer according to the rule

इष्टाहतस्वस्वहरेण युक्ते
ते वा भवेतां बहुधा गुणाप्ती ॥ †

The multiplier and the result become manifold when increased by the same multiple of their corresponding divisors.

* Līlāvatī, Chapter on *Kuṭṭaka*.
† *Ibid.*

Now we proceed to find an integer which will satisfy the three conditions of the problem. For this purpose also, we take the *kṣepas* to be positive. A value of the common multiplier satisfying the first two conditions has already been obtained. It is 4561. It should be multiplied by the dividend 8 of the third condition (which becomes the second condition when the second condition is regarded as the first). To the product 36488 add 9, the *kṣepa* of the third condition. The result 36497 is the *kṣepa* of a new problem, the divisor being 17 (the divisor of the third condition) and the dividend being 50096 (the product of the dividend 8 of the third condition and the divisor of the second condition, which is 6262 by the rule since there are three conditions in the problem). These *kṣepa*, divisor, and dividend give 5 as a multiplier. Multiply 5 by the divisor of the second condition which by the rule is 6262 and to the product add 4561 (the common multiplier satisfying the first two conditions, as obtained before). The result 35871 is the required integer when the *kṣepas* are all positive.

When the *kṣepas* are negative, the required integer

$$=6262\times 17-35871$$

$$=70583.$$

By adding to 35871 or 70583 (according as the *kṣepas* are positive or negative) any multiple of $62\times 101\times 17$ we can get as many values of the required integer as we please.

8. The *rationale* of the rule will be clear from the following:

Let the equations be

$$a_1x+c_1=b_1y \qquad \dots \quad (1)$$

$$a_2x+c_2=b_2z \qquad \dots \quad (2)$$

$$a_3x+c_3=b_3w \qquad \dots \quad (3)$$

Let $x=\alpha$ satisfy (1). Then $x=\alpha+b_1t$ (where t is any integer) will also satisfy (1). Let us suppose that this value of x also satisfies (2). Then we must have

$$a_2(\alpha+b_1t)+c_2=b_2z$$

or

$$a_2b_1t+(a_2\alpha+c_2)=b_2z.$$

This is the equation corresponding to the new problem referred to in the rule. If $t=\tau$ satisfy this equation, the value of x which satisfies (1) and (2) is $\alpha+b_1\tau$ as given by the rule.

If $x=\beta$ satisfy (1) and (2),

$$b_2(a_1\beta+c_1)=b_1b_2y$$

and

$$b_1(b_2\beta+c_2)=b_1b_2z$$

$$\therefore \quad (a_1b_2+a_2b_1)\beta+b_2c_1+b_1c_2=b_1b_2(y+z)$$

This shews that $x=\beta$ also satisfies the equation

$$(a_1b_2+a_2b_1)x+b_2c_1+b_1c_2=b_1b_2(y+z) \qquad \ldots \quad (4)$$

Hence the value of x, which satisfies (1), (2) and (3), also satisfies equations (4) and (3). Thus, the solution of the three equations (1), (2), and (3) is reduced to the solution of two simultaneous equations, *viz.*, (4) and (3). A solution of (4) has already been obtained, the value of x being $\alpha+b_1\tau$. From this we can find a value of x which will satisfy another equation also, *viz.*, equation (3). In this case the divisor in the first, *viz.* (4), of this new couple of equations (4) and (3), is b_1b_2 which is the product of the divisors in the first two equations. Hence the rule हरघ्नहारोऽत्र हरः प्रदिष्टः for three conditions in which the *kṣepas* are all positive. We are not concerned with the dividend and *kṣepa* in equation (4), as a solution has already been obtained without them.

When the *kṣepas* are negative, the equations become—

$$a_1x-c_1=b_1y \qquad \ldots \quad (5)$$

$$a_2x-c_2=b_2y \qquad \ldots \quad (6)$$

$$a_3x-c_3=b_3y \qquad \ldots \quad (7)$$

Equations (5) and (6) may be written as

$$a_1(b_1b_2-x)+c_1=b_1(a_1b_2-y),$$

$$a_2(b_1b_2-x)+c_2=b_2(a_2b_1-z),$$

i.e., as

$$a_1x'+c_1=b_1y' \qquad \ldots \quad (8)$$

$$a_2x'+c_2=b_2z' \qquad \ldots \quad (9)$$

where

$$x'=b_1b_2-x$$

Hence the solution of equations (5) and (6) can be made to depend on the solution of equations (8) and (9) which are the same as equations (1) and (2).

Because $$x' = b_1 b_2 - x,$$

$$\therefore \quad x = b_1 b_2 - x'.$$

Hence the rule for the case in which the *kṣepas* are negative. Similarly the rule for the solution of three equations with negative *kṣepas* might be deduced.

9. Now the question may be asked 'are the rule and the illustrative example Bhāskarāchārya's own, or are they later interpolations?'

The arguments against the genuineness of the rule may be stated as follows :

(*a*) The rule is not found in any other commentary of Līlāvatī, we have it.

(*b*) *Kuṭṭaka* has been treated both in Līlāvatī and also in Bījagaṇita, the two chapters being *nearly* identical. But the rule does not occur in any manuscript of Bījagaṇita, as handed down to us.

(*c*) In the chapter "Praśnádhyāya" of Siddhānta-Śiromani, Bhāskarāchārya has not used the rule uuder discussion but he has used all other kinds of *kuṭṭaka* that are found in his Bījagānita or in printed editions of Līlāvatī.

This last argument might at once be disposed of by saying that Līlāvatī contains many things not used in Siddhanta Śiromani.

The fact that Līlāvatī is much more widely known than Mahāvirāchārya's *Gaṇita-sāra-saṃgraha*—an earlier but better work from the point of view of a student of mathematics—might supply an answer to arguments (*a*) and (*b*). This shows that the majority of scholars cared more for astronomy than for mathematics. And Līlāvatī helped them to acquire the necessary amount of knowledge of mathematics in less time than *Gaṇita-sāra-saṃgraha*. As the rule in question must have been difficult in those days when the modern symbolical language of algebra was not invented and as it had no application in astronomy, readers might not be inclined to waste their time in learning and copying the rule and the illustrative example with their commentary Thus the rule with the example might have gradually disappeared from the manuscript copies of the text and commentaries of Līlāvatī as also of Bījagaṇita. The fact that chapters on *kuṭṭaka* in Līlāvatī and Bījagaṇita are not exactly identical goes to show that some omissions must have taken place in course of time. As the original manuscripts of the text and commentaries are not available, one is not justified in

denying the genuineness of the rule unless there be sufficient weightier evidence in one's favour.

10. Now we come to the arguments in favour of the genuineness of the rule. They may be stated as follows :

(*a*) The case was known to the earlier Hindu mathematicians also, as suggested by the word प्रदिष्ट. Mahāvīrāchārya who lived more than 250 years before Bhāskarāchārya has given a rule * for this case, different from the one given above, and nine examples † for exercise. Āryabhaṭa really considered a simple case of simultaneous indeterminate equations which may be stated thus :

$$x = b_1 y + c_1,\ x = b_2 z + c_2.$$

The general case in question is only an extension of this. It would not be wrong to suppose that this extension suggested itself also to other readers of Āryabhaṭa's works. In these circumstances it is only fair to say that the case occurred to Bhāskarāchārya also.

(*b*) Bhāskarāchārya has dealt with one case of simultaneous indeterminate equations (संश्लिष्टकुट्टक) in a stanza the wording ‡ of which strongly suggests that he must have considered the case in question without which the chapter would remain incomplete.

(*c*) The omission of the case is very difficult to explain unless, as is most unlikely, it be supposed that Bhāskarāchārya found the case too difficult for him and left it to be solved by a scholar of much less repute.

(*d*) The trend § of the commentary by Śrīdhara Mahāpātra shows that Bhāskarāchārya was the author of the rule which was included in the chapter on *kuṭṭaka*. If any illustrative examples did not occur in some of the copies of Līlāvatī he consulted, he made a note || to that effect. The absence of a similar note in connection with the case under consideration suggests that the case occurred in the copy of Gaṇeśa's commentary (as he had it), which he had consulted and criticised.

* *Gaṇita-sāra-saṃgraha*, stanza 115½.

† *Ibid*, stanzas 121½–129½.

‡ *Vide* § 2 above.

§ The commentator introduces the rule thus:

"एकस्मिन्नपि हारे बहुषु गुणेषु कुट्टकविधिना राशिज्ञानमभिधायाऽधुना बहुषु गुणकेषु बहुषु हारेषु च सत्सु राशिज्ञानार्थं संश्लिष्टबहुसामान्यकुट्टकमाह ।"

|| "इदमुदाहरणं क्वचित् क्वचित् पुस्तके न दृश्यते ।"

When he has given rules and examples not attributed by him to Bhāskarāchārya, he has indicated * the fact by naming their authors or otherwise.

11. The balance of arguments goes to support the genuineness of the rule and the example quoted above. We, therefore, conclude that Bhāskarāchārya considered the general case of simultaneous indeterminate equations of the first degree, which might be stated thus:

$$a_1x \pm c_1 = b_1y, \qquad a_2x \pm c_2 = b_2z,$$

$$a_3x \pm c_3 = b_3w,$$

$$\text{etc.} \qquad \text{etc.},$$

and that he gave the rule quoted above for its solution.

* "अथ ग्रन्थकृताऽनुक्तमपि केनचित् कृतमुदाहरणान्तरं दर्शयामः ।"
"अथ स्पष्टत्वात् ग्रन्थकृताऽनुक्तमपि द्वयोरपि ज्ञातमानयोरज्ञातवर्णयोर्वर्णज्ञानार्थं बालबोधाय सूत्रमुच्यते ।"
"अथ चापक्षेत्रे फलसाधनं श्रीधराचार्य आह ।"
"केशवोऽप्याह ।" "मम तातपादा अपि ।"
etc. etc. etc.

4

COLLISION OF α-PARTICLES WITH HELIUM ATOMS

BY

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[Read the 24th August, 1924]

Rutherford * has given a theory of the large angled scattering of α-particles by matter, based on the assumption that the atom consists of a concentrated positive charge Ne, surrounded by an equal number of electrons whose distances from the nucleus are large compared to the dimension of the latter. Experimental investigations by Geiger and Marsden † and lately by Chadwick ‡ have verified Rutherford's theory. In 1914, Darwin § extended Rutherford's theory to include the case of the collision of α-particles with the lighter atoms of matter in which the latter are set in motion by collision with α-particles. Darwin assumed (1) that the law of force between the nucleus and the α-particle is that of inverse square, (2) that during collision the nucleus and the α-particles behave as elastic spheres. Darwin was able to show that during a direct collision between an α-particle and hydrogen atom, the latter would move away with a velocity of $\frac{8}{5}$V, and its range would be about 4 times that of the colliding α-particle. Indirect experimental investigations by Marsden || showed the existence of H particles whose ranges were several times that of the α-particle. Using Wilson's method of photographing the tracks of ionising particles, Bose ¶ could directly photograph the tracks of the

* Rutherford, Phil. Mag., 1911, Vol. XXI, p. 669.

† Geiger and Marsden, Phil. Mag., 1913, Vol. XXV, p. 604.

‡ Chadwick, Phil. Mag., 1920, Vol. XL, p. 734.

§ Darwin, Phil. Mag., 1914, Vol. XXVII, p. 501.

|| Marsden, Phil. Mag., 1914, Vol. XXVII, p. 824.

¶ D. Bose, Zeit. f. Phys., 1922, Bd. 12, p. 207.

particles and show that in certain cases they are longer than the parent α-particle tracks. Darwin's theory of the scattering of the H particles was approximately verified by Bose.

For several reasons it is of interest to study photographically the collision of α-particles with Helium atoms. The mass of an α-particle is the same as that of a He atom and for such types of collision Darwin predicted that the angle of recoil between an α-particle and a colliding He atom would be either 0° or 90°. Further as Rutherford * has shown, the range of the recoiling He atom would be the same or four times that of the colliding α-particle according as the He atom is doubly or singly charged. On the assumption that an H particle is equivalent to a point charge, Chadwick and Bieler † have studied the close collision of α-particles with H particles. They have found that in order to explain the results of their experiment they have to suppose that He nucleus is an elastic oblate spheroid of semi-axes about 8×10^{-13} and 4×10^{-13} c. m. moving in the direction of its minor axis.

It is now generally assumed that a He atom consists of a doubly charged nucleus with two electrons, circulating round it. The orbits in which these electrons move are not definitely known but it is supposed that the two orbits are mutually inclined to one another. Recent experiments by Millikan and his pupils ‡ on the ionisation of various gas atoms by α-particles have shown that in ordinary gases the atom when ionised looses a single electron. Only in the case of Helium about 10% of the atoms are doubly ionised by collision with α-particles. This special behaviour of the He atom can be explained on the supposition that the two electrons in the He atom move in two crossed orbits.

In the following paper is described a photographic study of the collision of α-particles with He atoms, and we have obtained some important evidence bearing on the properties of the Helium atom which have been described above.

The apparatus designed for the experiment is practically the same as was used by Bose § in studying the collision of α-particles with Hydrogen, and also as that used by Bose and

* Rutherford, Phil. Mag., 1919, Vol. XXXVII, p. 571.

† Chadwick and Bieler, Phil. Mag., 1921, Vol. XLII, p. 923.

‡ Millikan, Gottschalk and Kelly, Phy. Rev., Vol. XV, 1920, p. 157. Wilkins, Phy. Rev., Vol. XIX, 1922, p. 210.

§ *Loc. cit.*

Ghosh * in photographing the tracks of recoiling rest atoms of Radioactive substance. The expansion chamber C (Fig. 1a), shewn elaborately in Fig. 16

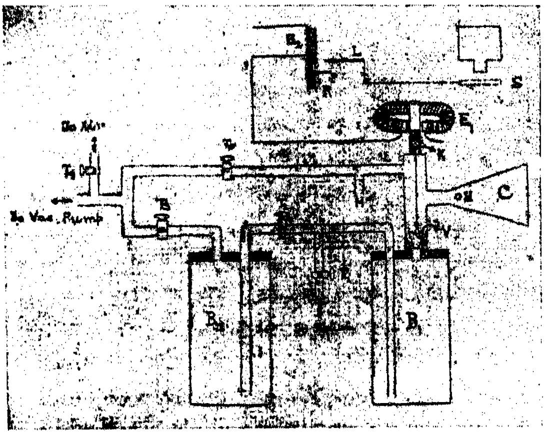


FIGURE 1a

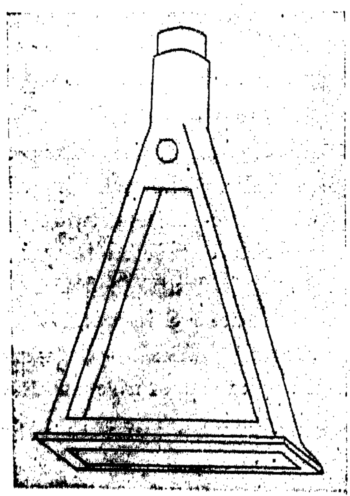


FIGURE 1b

consists of a shallow fan-shaped brass vessel, the top and front being replaced by glass. The length of the vessel is 15 c.m., maximum width 10 c.m, and the depth 2 c.m. It is specially designed to avoid the turbulent motion of the enclosed gas when it suddenly expands. A

* Bose and Ghosh, Phil. Mag., 1923, Vol. XLV, p. 1050.

copper wire with a deposit of Polonium is introduced into C, through the hole H. On suddenly expanding the gas in C, the supersaturated water vapour condenses on the tracks of the α-particles emitted from the wire at H thus rendering them visible. C is connected through the valve V to the graduated bottle B_1 which again through a glass tube is connected to B_2. B_1 and B_2 are partially filled with water, the level in B_1 being initially higher than that in B_2. In order to fill the chamber C with Helium gas, we proceed as follows. The taps T_1, T_2, T_3 and T_4 are closed and the vac. pump started. The chamber C and the spaces above the water level in B_1 are exhausted as indicated by the manometer M. T_5 is now closed and T_1 opened. Helium from the reservoir bubbles through the water in B_1 and fills the chamber C and the space above the water level in B_1 till the manometer shows atmospheric pressure, T_1 is then closed. The apparatus is now ready for work.

The method of expanding the gas in C is as follows. All the taps except T_2 and T_3 are closed and the pump set going. The water level rises in B_2 and comes down in B_1. When the level in B_1 has come down to the desired mark, T_2 is closed. The electromagnet E_1 is now excited. The soft iron core K attached to V is pulled up and thus opens the valve V. A portion of the gas enclosed in C rushes into B_1 causing an adiabatic expansion of the remaining gas in C. The supersaturated water vapour condenses on the ions in the tracks of α-particles from the source at H.

Sunlight focussed into the expansion chamber by a convex lens illuminates the ionisation tracks. The following arrangement is used for taking photograph of the α-particle tracks at the most suitable moment. The electromagnet E_2 connected in series with E_1 pulls a soft iron core upwards. The pin P attached to the core strikes against a lever L which releases the shutter S in front of the camera Z. The α-particle tracks are thus automatically photographed. The position of the pin P is adjusted by means of a screw R which can be moved up and down and fixed at any point. The shutter can thus be opened at the most suitable moment.

Results:—

(1) Altogether about 1,500 α-particle tracks are photographed. In one series of experiments out of a total number of 1,208 tracks 44 produce recoil atoms. The percentage of forking is about 3·65, a value which is rather high in comparison to what has been observed by Bose in Hydrogen.

(2) The range of the recoil atoms varies from 0·94 c.m. to 3·2 c.m. This value also appears abnormal compared to what Bose has observed in Hydrogen.

(3) The angles which the recoil atoms make with the tracks of α-particles after collision vary from 1° 23′ to 6° 17′ approximately. In no single case did we obtain any photograph in which the angle approached 90°. According to Darwin * both the values 0° and 90° are to be expected. Blackett † has taken a large number of photographs of α-particle tracks in a mixture of Helium and air, and amongst them are found a few tracks of recoil atoms which make angles approaching 90° with that of the colliding α-particles. These he has taken to be due to collision with Helium atoms. It seems to us that this is a rather unsatisfactory way of identifying the recoil He atoms.

So far as our results are concerned, we find that in the case of the long range recoil atoms, the angle of divergence approaches zero, while for short range recoil atoms it is larger. This behaviour can, we think, be satisfactorily explained, if we attribute like Chadwick and Bieler ‡ an oblate spheroidal structure to the α-particle. When the distance between an α-particle and a He nucleus is large, then both behave as point charges of equal masses and Darwin's theory can be applied to such collisions. But for very close collisions, both the nuclei orientate themselves so that their major axes are perpendicular to the line of mutual approach and then the recoiling atom is thrown forward in the direction of motion of the α-particle and the energy which is transferred to it is greater than what it would have received under the law of inverse square.

Finally we have not obtained any recoil atom track which is markedly greater than that of the parent α-particle. This would indicate that the He atoms do not exist as singly charged particles during the whole course of their existence as ionising agents. Recent investigations of Henderson § and Rutherford || seem to indicate that during the whole of its existence an α-particle can many times alter the charge it carries and only during the last few millimetres of its path does it carry a single charge.

* *Loc. cit.*

† Blackett, Proc. Roy. Soc. Lond., 1923, Vol. 103, p. 62.

‡ *Loc. cit.*

§ Henderson, Proc. Roy. Soc. Lond., 1922, Vol. 102, p. 496.

|| Rutherford, Proc. Camb. Phil. Soc., 1923, Vol. 21, p. 504.

Summary :—

With an apparatus specially designed for the purpose are obtained Wilson photographs of α-particle tracks in Helium.

Out of 1,203 lines the total number of forking is 44, the percentage being 3·65, a value which is rather high in comparison to that observed in Hydrogen.

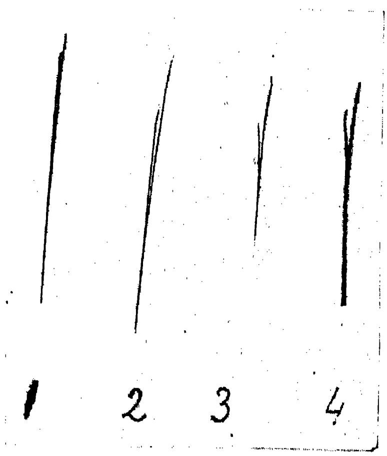
The range of recoil atoms varies from 0·94 c.m. to 3·2 c.m. This value also is high in comparison to that in H.

The angle of forking varies from 1° 23′ to 6° 17′ approximately. Contrary to what is expected from Darwin's theory, in no case was the angle of forking found to be near 90°. From the fact that almost all the recoil atoms move in the direction of the colliding α-particle, it is concluded that the He-nucleus is oblate spheroidal in shape.

No recoil atom track was found to be appreciably larger than those of the α-particles. This shows that the recoil atoms do not exist as singly charged particles during the whole course of their existence as ionising agents.

In conclusion I have much pleasure in recording my indebtedness to Dr. D. M. Bose, M.A., B.Sc., Ph.D., Ghose Prof. of Physics and my best thanks to Messrs. S. C. Datta, H. P. De and J. Das.

Bull. Cal. Math. Soc., Vol. XVII, Nos. 2 & 3.



RESULTS.

5

NEW METHODS IN EUCLIDEAN GEOMETRY OF FOUR DIMENSIONS.

1. INCLINATION OF PLANES

BY

R. C. BOSE

1. *Rectangular systems.*

Any tetrad of mutually orthogonal directed lines OA, OB, OC, OD emanating from a common point O, can be taken as axes of co-ordinates, OA as the axis of x_1, OB as the axis of x_2, OC as the axis of x_3, and OD as the axis of x_4. We will call such a system of axes of co-ordinates a *rectangular system* and denote it by O [ABCD].

Let O′[A′B′C′D′] be any other rectangular system. Let $l_{11}, l_{21}, l_{31}, l_{41}$; $l_{12}, l_{22}, l_{32}, l_{42}$; $l_{13}, l_{23}, l_{33}, l_{43}$; $l_{14}, l_{24}, l_{34}, l_{44}$ be the direction cosines of O′A′, O′B′, O′C′, O′D′ referred to O[ABCD]. If p'_1, p'_2, p'_3, p'_4 be the direction cosines of a line referred to O′[A′B′C′D′] and p_1, p_2, p_3, p_4 the direction cosines of the same line referred to O[ABCD], then evidently,

$$\left.\begin{aligned} p'_1 &= p_1 l_{11} + p_2 l_{21} + p_3 l_{31} + p_4 l_{41} \\ p'_2 &= p_1 l_{12} + p_2 l_{22} + p_3 l_{32} + p_4 l_{42} \\ p'_3 &= p_1 l_{13} + p_2 l_{23} + p_3 l_{33} + p_4 l_{43} \\ p'_4 &= p_1 l_{14} + p_2 l_{24} + p_3 l_{34} + p_4 l_{44} \end{aligned}\right\} \quad \ldots \quad (1)$$

Let $$\Delta \equiv \begin{vmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ l_{12} & l_{22} & l_{32} & l_{42} \\ l_{13} & l_{23} & l_{33} & l_{43} \\ l_{14} & l_{24} & l_{34} & l_{44} \end{vmatrix} \quad \ldots \quad (2)$$

Then $$\Delta^2 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

$$\therefore \quad \Delta = +1 \text{ or } -1$$

The rectangular system O[ABCD] is defined to be *congruent* to or *opposite* to the rectangular system O′[A′B′C′D′] according as Δ is $+1$ or -1.

If the rectangular system O[ABCD] *be congruent to (opposite to) the rectangular system* O′[A′B′C′D′] *then the latter is congruent to (opposite to) the former.*

This follows from the equality of the determinants

$$\begin{vmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ l_{12} & l_{22} & l_{32} & l_{42} \\ l_{13} & l_{23} & l_{33} & l_{43} \\ l_{14} & l_{24} & l_{34} & l_{44} \end{vmatrix} \text{ and } \begin{vmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \\ l_{41} & l_{42} & l_{43} & l_{44} \end{vmatrix}$$

The rectangular systems O′[A′B′C′D′] *and* O″[A″B″C″D″] *are congruent to one another if both are congruent to or both are opposite to another rectangular system* O[ABCD], *while they are opposite to one another if one is congruent to* O[ABCD] *while the other is opposite to it.*

Let l_{i1}, l_{i2}, l_{i3}, l_{i4}, $[i=1, 2, 3, 4]$ be the direction cosines of O′A′, O′B′, O′C′, O′D′ and m_{i1}, m_{i2}, m_{i3}, m_{i4} $[i=1, 2, 3, 4]$ the direction cosines of O″A″, O″B″, O″C″, O″D″ when referred to O[ABCD] as axes. If $p_{i1}, p_{i2}, p_{i3}, p_{i4}$ $[i=1, 2, 3, 4]$ be the direction cosines of O″A″, O″B″, O″C″, O″D″ referred to O′[A′B′C′D′], we have

$$p_{ij} = l_{1i}m_{1j} + l_{2i}m_{2j} + l_{3i}m_{3j} + l_{4i}m_{4j},$$

$$[i=1, 2, 3, 4 \,;\; j=1, 2, 3, 4] \quad \ldots \quad (3)$$

Let $\Delta'' \equiv \begin{vmatrix} p_{11} & p_{21} & p_{31} & p_{41} \\ p_{12} & p_{22} & p_{32} & p_{42} \\ p_{13} & p_{23} & p_{33} & p_{43} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{vmatrix}, \Delta' \equiv \begin{vmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{vmatrix}$

and let Δ be as in (2). Then from (3) we evidently have

$$\Delta'' = \Delta \Delta'.$$

Hence, $\Delta'' = +1$, if $\Delta = +1$, $\Delta' = +1$, or, $\Delta = -1$, $\Delta' = -1$

while $\Delta'' = -1$ if $\Delta = +1$, $\Delta' = -1$, or, $\Delta = -1$, $\Delta' = +1$

The two theorems proved above show that rectangular systems in four-space can be divided into two classes, such that

(*a*) any two members of the same class are congruent.

(*b*) any two members one from each class are opposite.

Rectangular systems obtained from O[ABCD] by an even number of interchanges of the letters A, B, C, D, are of the same class as O[ABCD] while systems obtained by an odd number of interchanges belong to the other class.

Let one of these classes be named class (A) and the other, the class (B). We shall then make it a convention to choose as frames of reference rectangular systems of a given class only, say class (A), and shall accordingly restrict ourselves to orthogonal transformations which change the axes from one rectangular system of the class (A) to another rectangular system of the same class. With this convention we can say that for any orthogonal transformation in which the axes are changed to lines whose direction cosines referred to the initial axes are $l_{i1}, l_{i2}, l_{i3}, l_{i4}$ $(i=1, 2, 3, 4)$ the determinant Δ given by (2) is always $+1$.

2. *Directed Planes.*

All the lines, planes and hyperplanes considered in this paper should be regarded as passing through a common point O even when express mention is not made of this fact.

In any plane we can conceive of two opposite rotational senses about the point O. A plane with a particular rotational sense associated with it will be called a *directed-plane*.

Let OP, OQ be two directed-lines. Let OP be denoted by p and OQ by q. Let a denote the plane POQ with a particular rotational sense associated with it, then $-a$ will denote the same plane with the opposite rotational sense associated with it.

We shall use the symbol $\widehat{pq}\,(a)$ to denote the angle θ through which p turns to occupy the position of q, θ being reckoned +ve or −ve

according as the rotation of p is in the same or the opposite sense to that associated with α and taken so as to satisfy the inequality

$$-\pi < \theta \lesseqgtr \pi .$$

We then have

$$\overset{\wedge}{pq}(\alpha) = -\overset{\wedge}{pq}(-\alpha) = -\overset{\wedge}{qp}(\alpha) = \overset{\wedge}{qp}(-\alpha)$$

The cosine of $\overset{\wedge}{pq}(\alpha)$ shall be denoted by the symbol (pq) and the sine of $\overset{\wedge}{pq}(\alpha)$ by the symbol $[pq, \alpha]$. It should be observed that $(pq) = (qp)$ while $[pq, \alpha] = -[qp, \alpha]$.

If l and l' be two directed-lines in α such that $\overset{\wedge}{ll'}(\alpha) = \pi/2$ we deduce from the trignometrical identity

$$\sin(A-B) \equiv \sin A \cos B - \cos A \sin B$$

the relation

$$[pq, \alpha] = (pl)(ql') - (pl')(ql) \qquad \dots \quad (4)$$

Let $O[LL'MM']$ be any rectangular system and let l, l', m, m' denote the directed-lines OL, OL′, OM, and OM′ respectively. Let $\lambda, \mu, \nu, \lambda', \mu', \nu'$ be directed-planes containing l, l'; l, m; l, m'; m, m'; m', l'; l', m respectively and with associated senses of rotation such that

$$\overset{\wedge}{ll'}(\lambda) = \overset{\wedge}{lm}(\mu) = \overset{\wedge}{lm'}(\nu) = \overset{\wedge}{mm'}(\lambda') = \overset{\wedge}{m'l'}(\mu') = \overset{\wedge}{l'm}(\nu') = \pi/2.$$

Then $\lambda, \mu, \nu, \lambda', \mu', \nu'$ are defined to be the (directed) co-ordinate planes of x_1x_2, x_1x_3, x_1x_4, x_3x_4, x_4x_2, and x_2x_3 respectively.

3. *Lemma. If p be any directed-line, and m, m', q directed-lines in a given directed-plane α, $\overset{\wedge}{mm'}(\alpha)$ being $\pi/2$, then*

$$(pq) = (pm)(qm) + (pm')(qm') \qquad \dots \quad (5)$$

Let m and m' be taken as the axes of x_1 and x_2 respectively, the axes of x_3 and x_4 being two other directed-lines l and l'.

The direction cosines of p are then (pm), (pm'), (pl), (pl') and the direction cosines of q are (qm), (qm'), 0, 0.

$$\therefore \quad (pq) = (pm)(qm) + (pm')(qm')$$

4. *Theorem. If* p, q, p', q' *be four directed lines, with direction cosines* p_i, q_i, p'_i, q'_i $[i=1, 2, 3, 4]$, then

$$[pq\,p'q'] \equiv \begin{vmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ p_1' & p_2' & p_3' & p_4' \\ q_1' & q_2' & q_3' & q_4' \end{vmatrix} \quad \ldots \quad (6)$$

remains unchanged by an orthogonal transformation.

Let $\qquad p_i, q_i, p'_i, q'_i \quad (i=1, 2, 3, 4)$

be the direction cosines of p, q, p', q' referred to O[ABCD] and r_i, s_i, r'_i, $s'_i (i=1, 2, 3, 4)$ be their direction cosines referred to O[A′B′C′D′]. Let l, l', m, m' denote the lines OA′, OB′, OC′, OD′ respectively and let l_i, l'_i, m_i, $m'_i (i=1, 2, 3, 4)$ be the direction cosines of these lines referred to O[ABCD], then

$$\begin{vmatrix} r_1 & r_2 & r_3 & r_4 \\ s_1 & s_2 & s_3 & s_4 \\ r'_1 & r'_2 & r'_3 & r'_4 \\ s'_1 & s'_2 & s'_3 & s'_4 \end{vmatrix} = \begin{vmatrix} (pl) & (pl') & (pm) & (pm') \\ (ql) & (ql') & (qm) & (qm') \\ (p'l) & (p'l') & (p'm) & (p'm') \\ (q'l) & (q'l') & (q'm) & (q'm') \end{vmatrix}$$

$$= \begin{vmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ p'_1 & p'_2 & p'_3 & p'_4 \\ q'_1 & q'_2 & q'_3 & q'_4 \end{vmatrix} \begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ l'_1 & l'_2 & l'_3 & l'_4 \\ m_1 & m_2 & m_3 & m_4 \\ m'_1 & m'_2 & m'_3 & m'_4 \end{vmatrix}$$

Since the second determinant on the left hand side is $+1$ according to the convention adopted, it is clear that $[pqp'q']$ remains unchanged by a transformation of rectangular axes (from a rectangular system of a given class to another of the same class).

5. *Theorem. If* p, p' *be directed lines in a given directed plane* α, *and* q, q' *directed lines in another directed plane* β, *then*

$$\mathrm{I}(\alpha, \beta) \equiv \frac{(pq)(p'q') - (pq')(p'q)}{[pp', \alpha][qq', \beta]} \quad \ldots \quad (7)$$

is invariant for all positions of p, p' q, q' *on the given planes.*

Let l, l' be two directed lines in α and m, m' two directed lines in β making $\widehat{ll'}(\alpha)=\pi/2$ and $\widehat{mm'}(\beta)=\pi/2$. Then by (5) we have

$$(pq)=(pm)(qm)+(pm')(qm')$$

$$(pq')=(pm)(q'm)+(pm')(q'm')$$

$$(p'q)=(p'm)(qm)+(p'm')(qm')$$

$$(p'q')=(p'm)(q'm)+(p'm')(q'm')$$

$$\therefore \quad (pq)(p'q')-(pq')(p'q)$$

$$=\{(pm)(p'm')-(pm')(p'm)\}\ \{(qm)(q'm')-(q'm)(qm')\}$$

$$=\{(mp)(m'p')-(m'p)(mp')\}[qq',\beta] \quad \text{by (4).}$$

Similarly

$$(mp)(m'p')-(m'p)(mp')=\{lm)(l'm')-(lm')(l'm)\}[pp',\alpha]$$

$$\therefore \quad \mathrm{I}(\alpha,\beta)=\frac{(pq)(p'q')-(pq')(p'q)}{[pp',\alpha][qq',\beta]}=(lm)(l'm')-(lm')(l'm).$$

Since l, l' are two arbitrary directed-lines in α making $\widehat{ll'}(\alpha)=\pi/2$ and m, m' are two arbitrary directed-lines in β making $\widehat{mm'}(\beta)=\pi/2$, by varying p, p', q, q' while keeping l, l', m, m' fixed we see that $\mathrm{I}(\alpha,\beta)$ is an invariant for all positions of p, p' in α, and q, q' in β.

6. *Theorem. If p, p' be directed lines in a given directed plane α and q, q' direccted lines in another directed plane β, then*

$$\mathrm{I}[\alpha,\beta]\equiv\frac{[pqp'q']}{[pp',\alpha][qq',\beta]} \quad \dots \quad (8)$$

is invariant for all positions of p, p' and q, q' on the given planes.

We have already seen that the value of $[pqp'q']$ is independent of the particular rectangular system chosen as the frame of reference (provided the choice be restricted by the convention adopted). Choose for the axes of x_1 and x_2 directed lines l and l' lying in α and making $\widehat{ll'}(\alpha)=\pi/2$ Let directed lines m and m' be axes of x_3 and x_4.

The direction cosines of p, p' are respectively,

$$(pl),(pl'),0,0 \quad \text{and} \quad (p'l),(p'l'),0,0$$

and the direction cosines of q and q' are respectively

$$(ql),(ql'),(qm),(qm') \quad \text{and} \quad (q'l),(q'l'),(q'm),(q'm')$$

Now
$$[pqp'q'] = \begin{vmatrix} (pl) & (pl') & 0 & 0 \\ (ql) & (ql') & (qm) & (qm') \\ (p'l) & (p'l') & 0 & 0 \\ (q'l) & (q'l') & (q'm) & (q'm') \end{vmatrix}$$

$$= -\begin{vmatrix} (pl) & (pl') \\ (p'l) & (p'l') \end{vmatrix} \begin{vmatrix} (qm) & (qm') \\ (q'm) & (q'm') \end{vmatrix}$$

$$= -[pp', \alpha]\{(qm)(q'm')-(qm')(q'm)\} \quad \ldots \text{ by (4).}$$

$$\therefore \quad \mathrm{I}[\alpha, \beta] = \frac{-\{(qm)(q'm')-(qm')(q'm)\}}{[qq', \alpha]}$$

This shows that $\mathrm{I}[\alpha, \beta]$ is independent of the position of p and p' in α. It can be similarly shown that $\mathrm{I}[\alpha, \beta]$ is independent of the position of q and q' in β.

7. *Planes simply perpendicular to two given planes.*

Two planes are defined to be *simply perpendicular* when they intersect in a line and each contains a line perpendicular to the other. A plane which is simply perpendicular to each of two given planes is said to be a *common perpendicular* to both.

Let α and β be two directed-planes. Let l, l' be directed-lines in α making $\widehat{ll'}(\alpha)=\pi/2$ and m, m' directed-lines in β making $\widehat{mm'}(\beta)=\pi/2$. Let p and q be directed-lines in α and β respectively. We shall determine the positions of p and q for which the plane P, which contains both p and q, is a common perpendicular plane to α and β.

Let p', q' be directed-lines in α and β respectively making $\widehat{pp'}(\alpha)=\pi/2$ and $\widehat{qq'}(\beta)=\pi/2$. When P is *simply perpendicular* to both α and β, p' and q' are each perpendicular to P and therefore $(pq')=(p'q)=0$. Again if $(pq')=(p'q)=0$ it is easy to see that P is *simply*

perpendicular to α and β. Hence the necessary and sufficient conditions that P is a common perpendicular plane to α and β are

$$(pq')=0, \qquad (p'q)=0.$$

Now
$$(pq')=(pm)(q'm)+(pm')(q'm'), \quad \text{by (5)}$$
$$=(ml)(pl)(q'm)+(ml')(pl')(q'm)$$
$$+(m'l)(pl)(q'm')+(m'l')(pl')(q'm') \quad \text{by (5)}$$

But $(p'l')=(pl)$, $(p'l)=-(pl')$, $(q'm')=(qm)$ and $(q'm)=(qm')$

$$\therefore \quad (pq')=-(ml)(pl)(qm')-(ml')(pl')(qm')$$
$$+(m'l)(pl)(qm)+(m'l')(pl')(qm)$$

Put $(pl)/(pl')=x$, $(qm)/(qm')=y$, $(ml)=a$, $(ml')=b$,

$$(m'l)=c, \ (m'l')=d$$

Then
$$\frac{(pq')}{(pl')(qm')}=-ax-b+cxy+dy \quad \ldots \quad (9)$$

Similarly
$$\frac{(p'q)}{(pl')(qm')}=-ay+bxy-c+dx \quad \ldots \quad (10)$$

$$\frac{(pq)}{(pl')(qm')}=axy+by+cx+d \quad \ldots \quad (11)$$

$$\frac{(p'q')}{(pl')(qm')}=a+bx+cy+dxy \quad \ldots \quad (12)$$

When $(pq')=(p'q)=0$, x and y must satisfy the simultaneous equations

$$b+ax-dy-cxy=0 \quad \ldots \quad (13)$$

$$c-dx+ay-bxy=0 \quad \ldots \quad (14)$$

Three cases may arise.

Case I. $a=b=c=d=0$

In this case the equations (13) and (14) are identically satisfied. p and q may be any lines whatsoever (through O) the first lying in α and the second in β. The planes α and β are absolutely perpendicular in this case.

Case II. $b/c=-a/d=-d/a=c/b$

i.e., either $$a=d, \qquad b=-c \qquad \ldots \quad (15)$$

or $$a=-d, \quad b=c \qquad \ldots \quad (16)$$

Equations (13) and (14) reduce to a single equation in this case, which when conditions (15) hold may be written

$$b(1+xy)+a(x-y)=0 \qquad \ldots \quad (17)$$

Corresponding to any arbitrary value of x we can always find one and only one value of y such that (13) and (14) are simultaneously satisfied. Thus for any position of p in α we can find a position of q in β, such that the plane containing p and q is a *common perpendicular* to α and β.

$$\text{Let } \widehat{pl}(\alpha)=\theta, \quad \widehat{qm}(\beta)=\phi$$

then, $$\widehat{pl'}(\alpha)=\frac{\pi}{2}+\theta, \qquad \widehat{qm'}(\beta)=\frac{\pi}{2}+\phi$$

$$\therefore \qquad x=(pl)/(pl')=-\cot\theta, \quad y=(qm)/(qm')=-\cot\phi.$$

Equation (17) may now be written

$$a(\tan\theta-\tan\phi)-b(1+\tan\theta\tan\phi)=0$$

$$\text{or} \quad \tan(\theta-\phi)=b/a \qquad \ldots \qquad \ldots \quad (18)$$

Equation (18) shows that in this case if the plane containing p and q is common perpendicular to α and β, then the plane containing r and s is also common perpendicular to α and β where r and s are any two directed lines in α and β respectively making

$$\widehat{pr}(\alpha)=\widehat{qs}(\beta).$$

Again equation (11) gives

$$(pq)=a \cos (\theta-\phi)+b \sin (\theta-\phi)=(rs).$$

Hence the acute angle cut out by α and β on any common perpendicular plane remains constant.

Similar results hold when equations (16) are satisfied.

Case III. Let neither of the conditions (15) or (16) be satisfied.

From equations (13) and (14) we derive the following:

$$(b^2-c^2)+(ab+cd)x-(bd+ac)y=0 \quad \ldots \quad (19)$$

$$(ab+cd)(x^2-1)-(a^2-b^2+c^2-d^2)x=0 \quad \ldots \quad (20)$$

$$(ac+bd)(y^2-1)+(a^2+b^2-c^2-d^2)y=0 \quad \ldots \quad (21)$$

If $\widehat{pl}(\alpha)$, $\widehat{qm}(\beta)$, x, y be taken as in Case II, the last two equations may be written

$$\tan 2\theta=\frac{2(ab+cd)}{a^2-b^2+c^2-d^2} \quad \ldots \quad \ldots \quad (22)$$

$$\tan 2\phi=\frac{2(ac+bd)}{-a^2-b^2+c^2+d^2} \quad \ldots \quad \ldots \quad (23)$$

The quadratic (20) determines two values of x or $-\cot\theta$, whose product is -1. Given any value of $\cot\theta$ satisfying (20), the directed-line p making $\widehat{pl}(\alpha)=\theta$ may be any one of two opposite directed-lines. The linear equation (19) gives a unique value of $\cot\phi$ corresponding to this value of $\cot\theta$ and thus determines q as one of two opposite dirceted-lnes. The plane containing p and q is then uniquely determined. The second root of (22) determines another common perpendicular plane in the same way. Hence we get in this case two planes common perpendicular to α and β. These planes are obviously absolutely perpendicular to one another (as the product of the two values of $\cot\theta$ is -1 and the same holds for $\cot\phi$).

8. *Angles between two directed-planes.*

Let α and β be any two directed planes through O. We have shown that we can always find at least two planes, absolutely perpendicular to one another and common perpendicular to α and β.

Let O [LL'MM'] be a rectangular system (of class A) such that the lines OL and OL' lie in one common perpendicular plane and the lines OM and OM' in the other. Let l, l', m, m' denote the directed lines OL, OL', OM, OM' respectively. Let γ and δ denote the (directed) co-ordinate plane of $x_1\ x_2$ and $x_3\ x_4$ respectively.

Let p be a directed-line common to α and γ, q a directed-line common to β and γ, p' a directed-line common to δ and α, and q' a directed-line common to δ and β, the lines being so selected as to make $\widehat{pp'}(\alpha)=\pi/2$; $\widehat{qq'}(\beta)=\pi/2$. Then $\widehat{pq}(\gamma)$ and $\widehat{p'q'}(\delta)$ are defined to be the angles between the directed planes α and β.

The rectangular system O[L'LM'M] satisfies the same conditions as O[LL'MM']. The (directed) co-ordinate planes of x_1x_2 and x_3x_4 for O[L'LM'M] are $-\gamma$ and $-\delta$. Hence the angles between α and β can also be taken to be $\widehat{pq}(-\gamma)$ and $\widehat{p'q'}(-\delta)$. Again if r and r' are directed lines opposite to p and p' respectively, they satisfy the same conditions as p and p'. So the angles between α and β may also be taken to be $\widehat{rq}(\gamma)$ and $\widehat{r'q'}(\delta)$ or $\widehat{rq}(-\gamma)$ and $\widehat{r'q'}(-\delta)$. If for example the angles between α and β are 15° and 135°, the angles may as well be taken to be −15° and −135°, −165° and −45° or 165° and 45°. It should be observed that the product of the cosines as well as the product of the sines of the two angles is unambiguosly determined and we proceed to show that these products are equal to $I(\alpha,\beta)$ and $I[\alpha,\beta]$ respectively.

When p, p', q, q' are chosen as above, we have

$$(pq')=(p'q)=0, \qquad [pp',\alpha]=[qq',\beta]=1$$

$$(pq)=\cos\theta_1, \qquad (p'q')=\cos\theta_2,$$

$$[pq,\gamma]=\sin\theta_1, \qquad [p'q',\delta]=\sin\theta_2$$

where θ_1 and θ_2 are the angles between α and β

$$I(\alpha,\beta)=\frac{(pq)(p'q')-(pq')(p'q)}{[pp',\alpha]\,[qq',\beta]}, \qquad \text{by } (7)$$

$$\therefore\quad \cos\theta_1\ \cos\theta_2=I(\alpha,\beta) \qquad \dots \quad (24)$$

Again the direction cosines of p, q, p', q' referred to O[LL′MM′] are respectively.

$$(pl), \quad (pl'), \quad 0, \quad 0;$$

$$(ql); \quad (ql'), \quad 0, \quad 0;$$

$$0, \quad 0, \quad (p'm), \quad (p'm');$$

and
$$0, \quad 0, \quad (q'm), \quad (q'm').$$

But $$\mathrm{I}[\alpha, \beta] = \frac{[pq\, p'q']}{[pp', \alpha][qq', \beta]}, \quad \text{by (8)}$$

$$= \{(pl)(ql') - (pl')(ql)\}\{(p'm)(q'm') - (p'm')(q'm)\}, \quad \text{by (6)}$$

$$= [pq, \gamma][p'q', \delta]$$

$$\therefore \quad \sin\theta_1 \quad \sin\theta_2 = \mathrm{I}[\alpha, \beta] \qquad \ldots \qquad \ldots \qquad (25)$$

From (24) and (25) we have

$$\cos(\theta_1 - \theta_2) = \mathrm{I}(\alpha, \beta) + \mathrm{I}[\alpha, \beta] \qquad \ldots \qquad \ldots \qquad (26)$$

$$\cos(\theta_1 + \theta_2) = \mathrm{I}(\alpha, \beta) - \mathrm{I}[\alpha, \beta] \qquad \ldots \qquad \ldots \qquad (27)$$

We shall make it a convention to take values of θ_1 and θ_2 satisfying

$$0 \leqq \theta_1 + \theta_2 \leqq \pi, \qquad 0 \leq \theta_1 - \theta_2 \leqq \pi \qquad \ldots \qquad \ldots \qquad (28)$$

With this convention the two angles between two planes are uniquely defined.

9. *Direction constants of directed planes.*

Let α be a given directed-plane. Let O[LL′MM′] be chosen as the frame of reference. Let l, l', m and m' denote the directed-lines OL, OL′, OM, OM′ respectively, and let λ, μ, ν, λ', μ', ν' denote the

directed co-ordinate planes of x_1x_2, x_1x_3, x_1x_4, x_3x_4, x_4x_2, and x_2x_3 respectively. Let

$$\left.\begin{aligned} a&=\mathrm{I}(\alpha,\lambda)-\mathrm{I}(\alpha,\lambda')\\ b&=\mathrm{I}(\alpha,\mu)-\mathrm{I}(\alpha,\mu')\\ c&=\mathrm{I}(\alpha,\nu)-\mathrm{I}(\alpha,\nu')\\ f&=\mathrm{I}(\alpha,\lambda)+\mathrm{I}(\alpha,\lambda')\\ g&=\mathrm{I}(\alpha,\mu)+\mathrm{I}(\alpha,\mu')\\ h&=\mathrm{I}(\alpha,\nu)+\mathrm{I}(\alpha,\nu') \end{aligned}\right\} \quad \dots \quad (29)$$

Then a, b, c, f, g, h are defined to be *direction constants* of α referred to O[LL'MM']. It should be observed that if a, b, c, f, g, h are the direction constants of α those of $-\alpha$ are $-a$, $-b$, $-c$, $-f$, $-g$, $-h$.

Let p and p' be two directed lines in α making $\widehat{pp'}(\alpha)=\phi$. Let p_i; p_i' $(i=1, 2, 3, 4)$ be the direction cosines of p and p'.

$$\mathrm{I}(\alpha,\lambda)=\frac{(pl)(p'l)-(pl')(p'l)}{\sin\phi}$$

$$\therefore\ \mathrm{I}(\alpha,\lambda)=(p_1p'_2-p'_1p_2)/\sin\phi$$

Similarly
$$\left.\begin{aligned} \mathrm{I}(\alpha,\mu)&=(p_1p'_3-p'_1p_3)/\sin\phi\\ \mathrm{I}(\alpha,\nu)&=(p_1p'_4-p'_1p_4)/\sin\phi\\ \mathrm{I}(\alpha,\lambda')&=(p_3p'_4-p'_3p_4)/\sin\phi\\ \mathrm{I}(\alpha,\mu')&=(p_4p'_2-p'_4p_2)/\sin\phi\\ \mathrm{I}(\alpha,\nu')&=(p_2p'_3-p'_2p_3)/\sin\phi \end{aligned}\right\} \quad \dots \quad (30)^*$$

* Ganguli defines the expressions on the left hand side of (30) to be the direction cosines of α. (Ganguli, *Analytical Geometry of Hyperspaces*, Vol. II Art 13).

Thus if a_1, b_1, c_1, f_1, g_1, h_1 be the direction consines of a plane whose direction constants are a, b, c, f, g, h then

$$a=a_1-f_1,\ b=b_1-g_1,\ c=c_1-h_1,\ f=a_1+f_1,\ g=b_1+g_1,\ h=c_1+h_1$$

These equations at once enable us to express the results obtained in this paper in terms of direction cosines but it will be found that in nearly all cases the use of direction constants enables us to express our formulae more compactly. The close analogy between formulae (33), (34), (37), (38), (46), (47) and formulae (62) to (75) with corresponding formulae of three diamensional geometry should also be noticed.

Now $\sin^2\phi=1-\cos^2\phi$

$$=(p_1^2+p_2^2+p_3^2+p_4^2)(p'^2_1+p'^2_2+p'^2_3+p'^2_4)-(p_1p'_1+p_2p'_2+p_3p'_3+p_4p'_4)^2$$

$$=(p_1p'_2-p'_1p_2)^2+(p_1p'_3-p'_1p_3)^2+(p_1p'_4-p'_1p_4)^2$$
$$+(p_3p'_4-p'_3p_4)^2+(p_4p'_2-p'_4p_2)^2+(p_2p'_3-p'_2p_3)^2 \quad \ldots \quad (31)$$

Again from the determinant identity

$$\begin{vmatrix} p_1 & p_2 & p_3 & p_4 \\ p'_1 & p'_2 & p'_3 & p'_4 \\ p_1 & p_2 & p_3 & p_4 \\ p'_1 & p'_2 & p'_3 & p'_4 \end{vmatrix}=0$$

it follows at once that

$$(p_1p'_2-p'_1p_2)(p_3p'_4-p'_3p_4)+(p_1p'_3-p'_1p_3)(p_4p'_2-p'_4p_2)$$
$$+(p_1p'_4-p'_1p_4)(p_2p'_3-p'_2p_3) \quad \ldots \quad (32)$$

From (29), (30), (31) and (32) we have

$$a^2+b^2+c^2=1 \quad \ldots \quad (33)$$

$$f^2+g^2+h^2=1 \quad \ldots \quad (34)$$

10 *To express the angles between two directed planes in terms of their direction constants.*

Let α and β be twe directed-planes with direction constants a, b, c, f, g, h, and a', b', c', f', g', h', respectively.

Let θ_1, θ_2 be the angles between them. Let p, p' be directed-lines in α, and q, q' directed lines in β, the direction consines of p, p', q, q' being p_i, p'_i, q_i, q'_i $(i=1, 2, 3, 4)$ respectively.

$$\cos\theta_1 \cos\theta_2=\mathrm{I}(\alpha, \beta) \text{ by } (24)$$

$$=\frac{\begin{vmatrix} (pq)(pq') \\ (p'q)(p'q') \end{vmatrix}}{[pp', \alpha][qq', \beta]} \text{ by } (7)$$

$$=\frac{\begin{vmatrix} p_1 & p_2 & p_3 & p_4 \\ p'_1 & p'_2 & p'_3 & p'_4 \end{vmatrix}\begin{vmatrix} q_1 & q_2 & q_3 & q_4 \\ q'_1 & q'_2 & q'_3 & q'_4 \end{vmatrix}}{[pp', \alpha][qq', \beta]}$$

Hence from (30)

$$\cos\theta_1 \cos\theta_2 = \mathrm{I}(\alpha, \lambda)\mathrm{I}(\beta, \lambda) + \mathrm{I}(\alpha, \mu)\mathrm{I}(\beta, \mu) + \mathrm{I}(\alpha, \nu)\mathrm{I}(\beta, \nu)$$

$$+\mathrm{I}(\alpha, \lambda')\mathrm{I}(\beta, \lambda') + \mathrm{I}(\alpha, \mu')\mathrm{I}(\beta, \mu') + \mathrm{I}(\alpha, \nu')\mathrm{I}(\beta, \nu') \quad \ldots \quad (35)$$

Again $\sin\theta_1 \sin\theta_2 = \mathrm{I}[\alpha, \beta]$ by (25)

$$= \begin{vmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ p'_1 & p'_2 & p_3 & p'_4 \\ q'_1 & q'_2 & q'_3 & q'_4 \end{vmatrix} / [pp', \alpha][qq', \beta]^*$$

Hence from (30)

$$-\sin\theta_1 \sin\theta_2 = \mathrm{I}(\alpha, \lambda)\mathrm{I}(\beta, \lambda') + \mathrm{I}(\alpha, \mu)\mathrm{I}(\beta, \mu') + \mathrm{I}(\alpha, \nu)\mathrm{I}(\beta, \nu')$$

$$+\mathrm{I}(\alpha, \lambda')\mathrm{I}(\beta, \lambda) + \mathrm{I}(\alpha, \mu')\mathrm{I}(\beta, \mu) + \mathrm{I}(\alpha, \nu')\mathrm{I}(\beta, \nu) \quad (36)$$

From (29), (35) and (36) we have

$$\cos(\theta_1 - \theta_2) = aa' + bb' + cc' \quad \ldots \quad (37)\dagger$$

$$\cos(\theta_1 + \theta_2) = ff' + gg' + hh' \quad \ldots \quad (38)\dagger$$

With the convention adopted in Art. 8, (37) and (38) uniquely determine θ_1 and θ_2, when a, b, c, f, g, h and a', b', c', f', g', h' are known.

Corollary (1).

$$\cos\theta_1 \cos\theta_2 = \tfrac{1}{2}(aa' + bb' + cc' + ff' + gg' + hh') \quad \ldots \quad (39)$$

$$\sin\theta_1 \sin\theta_2 = \tfrac{1}{2}(aa' + bb' + cc' - ff' - gg' - hh') \quad \ldots \quad (40)$$

* Ganguli, *Ana. Geo.* Part I, Page 31.

Ganguli obtains expressions for $\cos^2\theta_1 \cos^2\theta_2$ and $\sin^2\theta_1 \sin^2\theta_2$. From what has been proved it is evident that in extracting the square root the positive sign must be taken in the first and the negative sign in the second case.

† See foot-note page 117.

Corollary (2).

Since the direction constants of $-\alpha$ and $-\beta$ are respectively $-a$, $-b$, $-c$, $-f$, $-g$, $-h$ and $-a'$, $-b'$, $-c'$, $-f'$, $-g'$, $-h'$ the angles between $-\alpha$ and $-\beta$ are the same as those between α and β. Again the angles ϕ_1 and ϕ_2 between α and $-\beta$ (or between $-\alpha$ and β) are given by

$$\cos(\phi_1-\phi_2)=-aa'-bb'-cc'$$

$$\cos(\phi_1+\phi_2)=-ff'-gg'-hh'$$

$$\therefore \quad \phi_1+\phi_2=\pi-\theta_1-\theta_2$$

$$\phi_1-\phi_2=\pi-\theta_1+\theta_2$$

$$\therefore \quad \phi_1=\pi-\theta_1, \qquad \phi_2=-\theta_2.$$

Hence, if θ_1 and θ_2 be the angles between α and β; the angles between α and $-\beta$ (or between $-\alpha$ and β) are $\pi-\theta_1$ and $-\theta_2$.

Corollary (3). When α and β intersect in a line, we have a directed line r lying in both α and β. Let p', q' be directed-lines in α and β respectively.

$$\sin\theta_1 \sin\theta_2=\frac{[rr\,p'q']}{[rp',\alpha][rq',\beta]}, \qquad \text{by (25) and (8),}$$

$$=0, \qquad \text{by (6).}$$

$$\therefore \quad \cos(\theta_1+\theta_2)=\cos(\theta_1-\theta_2) \quad \text{or} \quad \theta_2=0.$$

Hence one of the angles between α and β vanishes.

Conversely when one of the angles between α and β vanishes, they intersect in a line.

(40) then shows that

The necessary and sufficient condition that α and β intersect in a line is

$$aa'+bb'+cc'=ff'+gg'+hh' \qquad \ldots \quad (41).*$$

* See foot-note page 117. Compare with Art. 17, *Ana. Geo.* Vol. II by Ganguli

The non-vanishing angle θ_1 between α and β is given by

$$\cos\theta_1 = aa'+bb'+cc' = ff'+gg'+hh', \qquad 0 \leqq \theta_1 \leqq \pi \quad \ldots \quad (42)$$

In case $\theta_1=0$, α and β become identical. In case $\theta_1=\pi$, β is the same as $-\alpha$. In other cases θ_1 cannot have the extreme values 0 and π.

Corollary (4). When α and β are simply perpendicular they intersect in a line, hence by the previous corollary

$$\sin\theta_1 \sin\theta_2 = 0$$

Again by definition α contains a directed line p perpendicular to β and β contains a directed line q perpendicular to α. Let p', q' be two other directed lines in α and β respectively.

Since p is perpendicular to β and q is perpendicular to α

$$(pq) = (pq') = (p'q) = 0$$

$$\therefore \quad \cos\theta_1 \cos\theta_2 = \frac{(pq)(p'q') - (p'q)(pq')}{[pp', \alpha][qq', \beta]} = 0$$

$$\therefore \quad \cos(\theta_1+\theta_2) = \cos(\theta_1-\theta_2) = 0$$

$$\therefore \quad \theta_1 = \pi/2, \qquad \theta_2 = 0.$$

i.e., The angles between two planes simply perpendicular to one another are $\pi/2$ and 0.

Hence from (39) and (40) the necessary and sufficient conditions that α and β may be simply perpendicular are

$$\left.\begin{aligned} aa'+bb'+cc' &= 0 \\ ff'+gg'+hh' &= 0 \end{aligned}\right\} \quad \ldots \quad (43)^*$$

Corollary (5).

$$\sin^2(\theta_1-\theta_2) = 1-(aa'+bb'+cc')^2 \quad \ldots \text{ by } (37)$$

$$= (a^2+b^2+c^2)(a'^2+b'^2+c'^2)-(aa'+bb'+cc')^2 \quad \ldots \text{ by } (33)$$

$$= (bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2$$

* See foot-note page 117. Compare with the condition of perpendicularity of two lines in three dimensional geometry.

As θ_1 and θ_2 satisfy (28) sin $(\theta_1-\theta_2)$ is positive. Hence

$$\sin(\theta_1-\theta_2)=\{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2\}^{\frac{1}{2}} \quad \ldots \quad (44)$$

Similarly

$$\sin(\theta_1+\theta_2)=\{(gh'-g'h)^2+(hf'-h'f)^2+(fg'-f'g)^2\}^{\frac{1}{2}} \quad \ldots \quad (45)$$

11. *Isocline Planes*

Let α and β be two directed-planes and let θ_1 and θ_2 be the angles between them. The angles between α and $-\beta$ are then by *Cor.* (2) to the last article, $\pi-\theta_1$ and $-\theta_2$.

α and β are defined to be *directly isocline in the* +ve *sense at an angle* ϕ *when* $\theta_1=\theta_2=\phi$ and to be *inversely isocline in the* +ve *sense at an angle* ϕ *when* $\pi-\theta_1=-\theta_2=\phi$. Again α and β are defined to be *directly isocline in the* −ve *sense at an angle* ϕ *when* $\theta_1=-\theta_2=\phi$ and to be *inversely isocline in the* −ve *sense at an angle* ϕ *when* $\pi-\theta_1=\theta_2=\phi$.

When α and β are isocline we can by the definition of angles between two planes find directed lines p, p' in α and q, q' in β so that $(pp')=(qq')=(pq')=(p'q)=0$ and $(pq)=(p'q')$ or $-(p'q')$ according as the isoclinism is direct or inverse. Hence the investigations of Case II Art. 7, apply to this case. α and β have an infinite number of common perpendicular planes one passing through each line of α (or β). The acute angle cut out by α and β on any common perpendicular plane remains constant.

Let the direction constants of α and β be a, b, c, f, g, h and a', b', c', f', g', h' respectively.

When α and β are directly isocline in the +ve sense $\theta_1-\theta_2=0$

$$\therefore \quad aa'+bb'+cc'=1, \qquad \text{by (37)}$$

Also $$a^2+b^2+c^2=a'^2+b'^2+c'^2=1, \qquad \text{by (33)}$$

$$\therefore \quad (a-a')^2+(b-b')^2+(c-c')^2=0,$$

or $$a-a'=b-b'=c-c'=0.$$

Again if $$a-a'=b-b'=c-c'=0,$$

$$\cos(\theta_1-\theta_2)=aa'+bb'+cc'$$

$$=a^2+b^2+c^2=1$$

$$\therefore \quad \theta_1=\theta_2.$$

Hence the necessary and sufficient conditions that α *and* β *may be directly isocline in the* $+$ve *sense are*

$$a-a'=b-b'=c-c'=0 \qquad \dots \quad (46)$$

Similarly

The necessary and sufficient conditions that (i) α *and* β *may be inversely isocline in the* $+$ve *sense* (ii) α *and* β *may be directly isocline in the* $-$ve *sense* (iii) α *and* β *may be inversely isocline in the* $-$ve *sense, are respectively*

$$(i) \quad a+a'=b+b'=c+c'=0 \qquad \dots \quad (47)$$

$$(ii) \quad f-f'=g-g'=h-h'=0 \qquad \dots \quad (48)$$

$$(iii) \quad f+f'=g+g'=h+h'=0 \qquad \dots \quad (49)$$

Again from (38) we find,

If α *and* β *are isocline in the* $+$ve *sense at an angle* ϕ

$$\cos 2\phi = \pm(ff'+gg'+hh'), \qquad 0 < 2\phi < \pi \qquad \dots \quad (50)$$

the upper or lower sign being taken according as the isoclinism is direct or inverse.

Similarly from (37),

If α *and* β *are isocline in the* $-$ve *sense at an angle* ϕ

$$\cos 2\phi = \pm(aa'+bb'+cc'), \qquad 0 < 2\phi < \pi \qquad \dots \quad (51)$$

the upper or lower sign being taken according as the isoclinism is direct or inverse.

It follows from Cor. (2) to the last article that if α and β are directly (inversely) isocline in a given sense at an angle ϕ, $-\alpha$ and $-\beta$ are directly (inversely) isocline in the same sense at an angle ϕ, and α and $-\beta$ or $-\alpha$ and β are inversely (directly) isocline in the same sense at an angle ϕ. This enables us to define isoclinism between two planes without any reference to the rotational senses attached to them.

Let P and Q be two planes. Let the directed-planes γ and δ, obtained by associating particular rotational senses with P and Q be $+$ve-*ly* ($-$ve-*ly*) isocline at an angle ψ. The directed planes obtained by reversing the rotational senses with either or both of P and Q are $+$ve-*ly* ($-$ve-*ly*) isocline at the same angle *ie* $-\gamma$ and δ, γ and $-\delta$ or $-\gamma$ and $-\delta$ are also isocline at an angle ψ though the iso-

clinism may be direct or inverse. P and Q may then be called +ve-*ly* (−ve-*ly*) isocline at an angle ψ. Thus

Two planes P and Q will be called +ve-*ly* (−ve-*ly*) *isocline at an angle ψ, if the directed planes obtained by associating arbitrary rotational senses with P and Q are always* +ve-*ly* (−ve-*ly*) *isocline at an angle ψ.*

Corollary (1). *If directed planes α and β are both directly or both inversely isocline to another directed plane γ in the* +ve (−ve) *sense, then α and β are directly isocline in the* +ve(−ve) *sense.*

If the directed-plane α is direetly isocline, and the directed-plane β inversely isocline to another directed-plane γ in the +ve (−ve) *sense, then α and β are inversely isocline in the* +ve (−ve) *sense.*

This follows at once from conditions (46) to (49). Without reference to rotational sense the above may be stated as follows :—

If planes P and Q are each isocline to a given plane R in the +ve (−ve) sense, then P and Q are isocline in the +ve (−ve) sense.*

Corollary (2). *If a directed plane α intersects in a line each of two directed-planes β and γ directly (inversely) isocline to one another in any sense, the non-vanishing angles which α makes with β and γ are equal (supplementary).*

To fix ideas let β and γ be +ve-*ly* isocline. Let the direction constants of β be a, b, c, f, g, h and of γ be $\pm a, \pm b, \pm c, f', g', h'$, the upper or lower signs being taken according as the isoclinism is direct or inverse. Let the direction constants of α be $a_1, b_1, c_1, f_1, g_1, h_1$. If θ and ϕ be the non-vanishing angles which α makes with β and γ, we have from (42).

$$\cos\theta = aa_1 + bb_1 + cc_1 \qquad , \quad 0 < \theta < \pi$$

$$\cos\phi = \pm(aa_1 + bb_1 + cc_1) \qquad , \quad 0 < \phi < \pi,$$

the upper or lower sign being taken according as β and γ are directly or inversely isocline.

In the first case $\theta = \phi$, while in the second case $\theta = \pi - \phi$.

Without reference to rotational senses this corollary may be stated as follows :

A plane intersecting in lines two given isocline planes makes equal acute angles with both.†

* Manning, *Geometry of Four Dimensions*, p. 191, Art. 109, *Th.*

† Stringham, "*On the Geometry of Planes in a Parabolic Space of Four Dimensions*" Transactions of the American Mathematical Society, Vol. 2, p. 210, Art. 30 (2).

Also Manning, *Loc. cit.*, p. 194, Art. 111, *Th.* 1.

Corollary (3). *If directed-planes γ and δ vary so that γ remains directly (or inversely) isocline in the* $+$ve *sense to a fixed directed-plane α, and δ remains directly (or inversely) isocline to a fixed directed plane β, and if θ_1 and θ_2 be the angles between γ and δ, then $\theta_1-\theta_2$ remains constant.*

Let $a_1, b_1, c_1, f_1, g_1, h_1$ and $a_2, b_2, c_2, f_2, g_2, h_2$ be the direction constants of α and β. Let the direction constants of γ for a particular position be $a_3, b_3, c_3, f_3, g_3, h_3$ and the direction constants of δ for a particular position be $a_4, b_4, c_4, f_4, g_4, h_4$.

If γ remains directly isocline to α in the $+$ve sense, and δ remains directly isocline to β in the $+$ve sense, we have by (46)

$$a_3=a_1, \quad b_3=b_1, \quad c_3=c_1\,; \quad a_4=a_2, \quad b_4=b_2, \quad c_4=c_2.$$

Hence from (37) $\theta_1-\theta_2$ is constant and is given by

$$\cos(\theta_1-\theta_2)=a_1a_2+b_1b_2+c_1c_2 \qquad 0 \lesseqgtr \theta_1-\theta_2 \lesseqgtr \pi$$

We may similarly treat the cases when (*i*) γ remains directly isocline to α and δ inversely isocline to β (*ii*) γ remains inversely isocline to α and δ directly isocline to β (*iii*) γ remains inversely isocline to α and δ inversely isocline to β, the isoclinism being in the $+$ve sense in every case.

If directed-planes γ and δ vary so that γ remains directly (or inversely) isocline in the $-$ve sense to a fixed directed plane α and δ remains directly (or inversely) isocline in the $-$ve sense to a fixed directed plane β, and if θ_1 and θ_2 be the angles between γ and δ then $\theta_1+\theta_2$ remains constant.

Corollary (4). *If planes P and Q are isocline to a given plane R at the same angle but in opposite senses, P and Q intersect in a line.*‡

Let P be positively and Q negatively isocline to R at an angle ϕ. Let γ be the directed-plane obtained by associating a particular rotational sense with R. Let α and β be directed-planes obtained by associating rotational senses with P and Q such that each of α and β is directly isocline to γ. Let a, b, c, f, g, h be the direction constants of γ; $a_1, b_1, c_1, f_1, g_1, h_1$ the direction constants of α and $a_2, b_2, c_2, f_2, g_2, h_2$ the direction constants of β. From (46) and (48)

$$a=a_1, \quad b=b_1, \quad c=c_1\,; \quad f=f_2, \quad g=g_2, \quad h=h_2$$

‡ Manning, *Loc. Cit.*, p. 192, Art. 109 *Cor.*

From (50) and (51)

$$\cos 2\phi = aa_2 + bb_2 + cc_2 = ff_1 + gg_1 + hh_1$$

Hence $$a_1a_2 + b_1b_2 + c_1c_2 = f_1f_2 + g_1g_2 + h_1h_2,$$

which by *Cor.* (3) Art. 10 is the condition that α and β intersect in a line. Hence P and Q intersect in a line.

12. *Direction constants of directed-planes absolutely perpendicular to each other.*

When two directed-planes are absolutely perpendicular the angles between them may be $\pi/2$, $\pi/2$ or $\pi/2$, $-\pi/2$.

In the first case the planes are directly isocline in the +ve sense and inversely isocline in the −ve sense. They will be called absolutely perpendicular in the +ve sense.

In the second case the planes are directly isocline in the −ve sense and inversely isocline in the +ve sense. They will be called absolutely perpendicular in the −ve sense.

We shall denote by α' the directed plane absolutely perpendicular to the directed-plane α in the +ve sense, and by $-\alpha'$ the directed plane absolutely perpendicular to α in the −ve sense.

If the direction constants of α be a, b, c, f, g, h conditions (46) to (49) show that, the direction constants of α' are $\quad a, b, c, -f, -g, -h$

and the direction constants of $-\alpha'$ are $\quad -a, -b, -c, f, g, h.$

It should be observed that $-\alpha'$ is absolutely perpendicular to $-\alpha$ in the +ve sense, since the direction constants of $-\alpha$ are $-a, -b, -c, -f, -g, -h$.

Corollary (1). *If a directed-plane is directly (inversely) isocline to a given directed-plane α in the* +ve *sense, it is directly (inversely) isocline to α' in the same* sense. *If a directed-plane is directly (inversely) isocline to α in the* −ve *sense it is inversely (directly) isocline to α' in the same sense.*

The proof is obvious.

Corollary (2). *If the angles between the directed-planes α and β be θ_1 and θ_2, the angles between α' and β are $(\pi/2-\theta_1)$ and $(\pi/2-\theta_2)$.*

Let a, b, c, f, g, h; a', b', c', f', g', h' be the direction-constants of α and β respectively. Then the direction constants of α' are

$a, b, c, -f, -g, -h$. If ϕ_1 and ϕ_2 be the angles between a' and β we have by (37) and (38).

$$\cos(\theta_1-\theta_2)=aa'+bb'+cc'=\cos(\phi_1-\phi_2)$$

$$\cos(\theta_1+\theta_2)=ff'+gg'+hh'=-\cos(\phi_1+\phi_2)$$

$$\therefore \quad \theta_1-\theta_2=\phi_1-\phi_2 \quad \text{and} \quad \theta_1+\theta_2=\pi-(\phi_1+\phi_2)$$

$$\therefore \quad \phi_1=(\pi/2-\theta_1), \quad \phi_2=(\pi/2-\theta_2)$$

Corollary (3). *Given any two directed planes α and β, we can always find eight directed planes $\gamma, -\gamma, \gamma', -\gamma', \delta, -\delta, \delta', -\delta'$ each of which is isocline to α and to β in opposite senses.*

Let the direction constants of α and β be a, b, c, f, g, h and a', b', c, f', g', h' respectively. Let γ and δ be the directed-planes with direction constants a, b, c, f', g', h' and a', b', c', f, g, h respectively. We can then at once write down the direction constants of $-\gamma$, $-\delta$, γ', δ', $-\gamma'$ and $-\delta'$. Conditions (46) to (49) then show that each of $\gamma, -\gamma, \gamma', -\gamma', \delta, -\delta, \delta', -\delta'$ is isocline to α and β in opposite senses.

Without reference to rotational senses we can write the above in the following way :—

*Given any two planes P and Q, we can always find planes A, A', B, B', each of which is isocline to P and Q in opposite senses, A and A' being absolutely perpendicular to one another and the same holding for B and B'.**

13. *Conditions that a given line lies in a given plane.*

Let α be a directed-plane with direction constants a, b, c, f, g, h. Let q be a directed-line with direction cosines q_1, q_2, q_3, q_4. Required to find the conditions that q lies in α.

Let p, p' be directed-lines in α, with direction cosines p_i ; p'_i ($i=1, 2, 3, 4$) and making $\widehat{pp'}(\alpha)=\pi/2$.

If q lies in α and makes $\widehat{pq}(\alpha)=\psi$, we have

$$q_i=p_i\cos\psi+p'_i\sin\psi \qquad (i=1, 2, 3, 4) \qquad \text{... by (5)}$$

Eliminating $\cos\psi$ and $\sin\psi$

$$\left\|\begin{matrix} q_1 & q_2 & q_3 & q_4 \\ p_1 & p_2 & p_3 & p_4 \\ p'_1 & p'_2 & p'_3 & p'_4 \end{matrix}\right\|=0 \qquad \text{... (52)}$$

* Manning proves this theorem only for the particular case when P and Q intersect in a line. *Loc. Cit.*, p. 187 Art. 107 *Th.* 2.

Let $\lambda, \mu, \nu, \lambda', \mu', \nu'$ be the (directed) co-ordinate planes of reference as in Art. 9. Since $\widehat{pp'}(a)=\pi/2$ we have

$$(p_1p_2'-p_2p_1')=\mathrm{I}(a, \lambda) \qquad \text{by (30)}$$

$$=\tfrac{1}{2}(f+a) \qquad \text{by (29).}$$

Similarly,

$$(p_1p'_3-p_1'p_3)=\tfrac{1}{2}(g+b), \; (p_1p'_4-p_1'p_4)=\tfrac{1}{2}(h+c)$$

$$(p_3p_4'-p_4p'_3)=\tfrac{1}{2}(f-a), \quad (p_4p'_2-p'_4p_2)=\tfrac{1}{2}(g-b)$$

and $$(p_2p'_3-p_3p'_2)=\tfrac{1}{2}(h-c)$$

Conditions (52) may now be written

$$\left.\begin{aligned}(f-a)q_2+(g-b)q_3+(h-c)q_4&=0\\ (f-a)q_1-(h+c)q_3+(g+b)q_4&=0\\ (g-b)q_1+(h+c)q_2-(f+a)q_4&=0\\ (h-c)q_1-(g+b)q_2+(f+a)q_3&=0\end{aligned}\right\} \quad \ldots \quad (53)$$

These are the necessary and sufficient conditions that q lies in a. Only two of these conditions are independent for any two being given the other two readily follow by using the identical relation,

$$(f+a)(f-a)+(g+b)(g-b)+(h+c)(h-c)=0.$$

The above conditions may be written in a different form. Multiplying the equations in (53) by $q_2, q_1, -q_4$ and q_3 respectively and adding,

$$f=(q_1^2+q_2^2-q_3^2-q_4^2)a+2(q_2q_3-q_1q_4)b+2(q_2q_4+q_1q_3)c.$$

We can similarly deduce that

$$f=aQ_1+bQ_2+cQ_3, \quad g=aQ'_1+bQ'_2+cQ'_3,$$

$$h=aQ''_1+bQ''_2+cQ''_3 \quad \ldots \quad (54)$$

$$a=fQ_1+gQ'_1+hQ''_1, \quad b=fQ_2+gQ'_2+hQ''_2,$$

$$c=fQ_3+gQ'_3+hQ''_3 \quad \ldots \quad (55)$$

where

$$
\left.\begin{array}{ll}
Q_1=q_1^2+q_2^2-q_3^2-q_4^2, & Q_2=2(q_2q_3-q_1q_4), \\
 & Q_3=2(q_2q_4+q_1q_3) \\
Q'_1=2(q_2q_3+q_1q_4), & Q'_2=(q_1^2-q_2^2+q_3^2-q_4^2), \\
 & Q'_3=2(q_3q_4-q_1q_2) \\
Q''_1=2(q_2q_4-q_1q_3), & Q''_2=2(q_3q_4+q_1q_2), \\
 & Q''_3=(q_1^2-q_2^2-q_3^2+q_4^2)
\end{array}\right\} \quad (56)
$$

*Corollary. Through any line q not lying in a directed-plane β we can always pass one and only one directed-plane isocline to β in a given way.**

Let a, b, c, f, g, h be the direction constants of β and q_1, q_2, q_3, q_4 the direction cosines of q, when the directed-plane γ whose direction constants are

$$a, b, c, \quad aQ_1+bQ_2+cQ_3, \quad aQ'_1+bQ'_2+cQ'_3, \quad aQ''_1+bQ''_2+cQ''_3$$

is directly isocline to β in the +ve sense and the directed-plane δ whose direction-constants are

$$fQ_1+gQ'_1+hQ''_1, \qquad fQ_2+gQ'_2+hQ''_2,$$

$$fQ_3+gQ'_3+hQ''_3, \qquad f, g, h$$

is directly isocline to β in the $-$ve sense, where $Q_1, Q_2, Q_3, Q'_1, Q'_2, Q'_3, Q''_1, Q''_2, Q''_3$ are functions of q_1, q_2, q_3, q_4 given by (56).

Again $-\gamma$ and $-\delta$ also pass through q and are inversely isocline to β in the +ve and $-$ve senses respectively.

14. *Direction constants of common perpendicular planes to two given planes.*

Let α and β be two directed-planes with direction constants a, b, c, f, g, h and a', b', c', f', g', h' respectively. Let u, v, w, x, y, z be the

† Strigham, *Loc. Cit.* p. 210, Art. 80(3).
and Manning, *Loc. Cit.*, p. 186, Art. 107, *Th.* I.

direction constants of any plane simply perpendicular to each of α and β. Then by (33), (34) and (43)

$$au+bv+cw=0 \qquad \dots \quad (57)\text{i}$$

$$a'u+b'v+c'w=0 \qquad \dots \quad (57)\text{ii}$$

$$fx+gy+hz=0 \qquad \dots \quad (57)\text{iii}$$

$$f'x+g'y+h'z=0 \qquad \dots \quad (57)\text{iv}$$

$$u^2+v^2+w^2=1 \qquad \dots \quad (57)\text{v}$$

$$x^2+y^2+z^2=1 \qquad \dots \quad (57)\text{vi}$$

Case I. Let α aad β be not isocline and let θ_1, θ_2 be the angles between them.

From (57)i, (57)ii

$$\frac{u}{(bc'-b'c)}=\frac{v}{(ca'-c'a)}=\frac{w}{(ab'-a'b)}=k \quad \text{(say)}$$

and from (57) iii, (57)iv

$$\frac{x}{(gh'-g'h)}=\frac{y}{(hf'-h'f)}=\frac{z}{(fg'-f'g)}=k' \quad \text{(say)}$$

But

$$k^2=\frac{u^2+v^2+w^2}{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}$$

$$=\frac{1}{\sin^2(\theta_1-\theta_2)} \qquad \dots \quad (44)$$

$$\therefore \quad k=\pm\frac{1}{\sin(\theta_1-\theta_2)},$$

similarly
$$k'=\pm\frac{1}{\sin(\theta_1+\theta_2)}.$$

Thus the four sets of values of u, v, w, x, y, z satisfying

$$\left.\begin{aligned}\frac{u}{(bc'-b'c)}=\frac{v}{(ca'-c'a)}=\frac{w}{(ab'-a'b)}=\pm\frac{1}{\sin(\theta_1-\theta_2)}\\ \frac{x}{(gh'-g'h)}=\frac{y}{(hf'-h'f)}=\frac{z}{(fg'-f'g)}=\pm\frac{1}{\sin(\theta_1+\theta_2)}\end{aligned}\right\}\quad(58)$$

give the six direction constants of the four directed planes simply perpendicular to each of α *and* β. *If* γ *is one of these planes the other three are* $-\gamma$, γ' *and* $-\gamma'$.

Case II. Let α *and* β *be isocline in the* +ve *sense at angle* $\theta\neq\pi/2$.

Then $a=a'$, $b=b'$, $c=c'$ or $a=-a'$, $b=-b'$, $c=-c'$ according as the isoclinism is direct or inverse. The equations (57)i and (57)ii become identical.

Hence *any set of values of* u, v, w, x, y, z *satisfying*

$$au+bv+cw=0,\quad u^2+v^2+w^2=1$$

$$\frac{x}{(gh'-g'h)}=\frac{y}{(hf'-h'f)}=\frac{z}{(fg'-f'g)}=\pm\frac{1}{\sin 2\theta}$$

gives the six direction constants of a directed-plane, common perdendicular to α *and* β.

Case III. Let α *and* β *be isocline in the* —ve *sense at an angle* $\phi\neq\pi/2$.

Then $f=f'$. $g=g'$, $h=h'$, or $f=-f'$, $g=-g'$, $h=-h'$, according as the isoclinism is direct or inverse. The equations (57)iii and (57)iv become identical.

Hence any set of values of u, v, w, x, y, z satisfying

$$fx+gy+hz=0\qquad x^2+y^2+z^2=1$$

$$\frac{u}{(bc'-b'c)}=\frac{v}{(ca'-c'a)}=\frac{w}{(ab'-a'b)}=\pm\frac{1}{\sin 2\phi}$$

gives the six direction constants of a directed-plane common perpendicular to α *and* β.

Case IV. Let α and β be absolutely perpendicular.

Then $a=\pm a'$, $b=\pm b'$, $c=\pm c'$, $f=\mp f'$, $g=\mp g'$, $h=\mp h'$, the upper or lower signs being taken according as α and β are absolutely perpendicular in the +ve or the −ve sense. Equations (57) i and (57)ii become identical and so do equations (57) iii and (57) iv. The conditions that a directed-plane γ is simply perpendicular to α are also the conditions that γ is simply perpendicular to β. Hence any directed plane simply perpendicular to α is also simply perpendicular to β.

Any set of values of u, v, w, x, y, z *satisfying*

$$\left.\begin{aligned} &au+bv+cw=0, \qquad u^2+v^2+w^2=1 \\ &fx+gy+hz=0, \qquad x^2+y^2+z^2=1 \end{aligned}\right\} \qquad \ldots \qquad (61)$$

give the six direction constants of a directed plane common perpendicular to α and β.

*Corollary. If directed-planes α and β are each isocline to a given directed-plane γ in opposite senses, we can always find directed planes $\delta, -\delta, \delta', -\delta'$ each of which is simply perpendicular to each of α, β and γ.**

Suppose the direction constants of γ are a, b, c, f, g, h. The direction constants of α must be of the form $\pm a, \pm b, \pm c, f_1, g_1, h_1$, if α and γ are +ve-ly isocline, and those of β must be of the form $a_1, b_1, c_1, \pm f, \pm g, \pm h$, if β and γ are −ve-ly isocline, the upper or lower signs being taken in each case according as the isoclinism is direct or inverse. The necessary and sufficient conditions that a directed-plane with direction-constants u, v, w, x, y, z may be simply perpendicular to each of α, β, γ are by (43)

$$\begin{aligned} &au+bv+cw=0, \qquad a_1u+b_1v+c_1w=0 \\ &fx+gy+hz=0, \qquad f_1x+g_1y+h_1z=0 \end{aligned}$$

Hence the four sets of values of u, v, w, x, y, z satisfying

$$\left.\begin{aligned} \frac{u}{(bc_1-b_1c)}&=\frac{v}{(ca_1-c_1a)}=\frac{w}{(ab_1-a_1b)}=\pm\frac{1}{\sin 2\theta_1} \\ \frac{x}{(gh_1-g_1h)}&=\frac{y}{(hf_1-h_1f)}=\frac{z}{(fg_1-f_1g)}=\pm\frac{1}{\sin 2\theta_2} \end{aligned}\right\}$$

give the six direction constants of the four directed-planes simply perpendicular to each of α, β, γ, where θ_1 is the angle of isoclinism between β and γ and θ_2 the angle of isoclinism between α and γ. If δ be one of these planes the other three are obviously $-\delta$, δ' and $-\delta'$.

* Manning, *Loc. Cit.*, p. 189, Art. 103, *Th.* 2.

15. *Triads of directed planes.*

Let α, β, γ be three directed-planes with direction constants

$$a_1, \quad b_1, \quad c_1, \quad f_1, \quad g_1, \quad h_1,$$

$$a_2, \quad b_2, \quad c_2, \quad f_2, \quad g_2, \quad h_2$$

$$a_3, \quad b_3, \quad c_3, \quad f_3, \quad g_3, \quad h_3$$

Then α, β, γ are said to form a triad of directed-planes. This triad will be called a *regular triad* if no two of α, β, γ are isocline. If α, β, γ be considered to be the primary triad then α', β', γ' form the *complementary triad.*

Again let λ be the directed plane with direction constants $u_1, v_1, w_1, x_1, y_1, z_1$ given by

$$\frac{u_1}{(b_2c_3-b_3c_2)}=\frac{v_1}{(c_2a_3-c_3a_2)}=\frac{w_1}{(a_2b_3-a_3b_2)}=\frac{1}{\sin(\theta_1-\theta'_1)}$$

$$\frac{x_1}{(g_2h_3-g_3h_2)}=\frac{y_1}{(h_2f_3-h_3f_2)}=\frac{z_1}{(f_2g_3-f_3g_2)}=\frac{1}{\sin(\theta_1-\theta'_1)}$$

where θ_1 and θ'_1 are the angles between β and γ. λ is then defined to be the *polar plane* of the pair of directed planes β, γ. It is to be noticed that λ is one of the directed-planes simply perpendicular to β and γ, and also that the polar plane of the pair γ, β is not λ but $-\lambda$. Similarly let the directed-planes u, v be the polar planes of the pairs γ, α and α, β respectively. Then the triad λ, μ, ν shall be called the *polar triad* of α, β, γ (It is understood that the triad α, β, γ is regular). It can then be easily shown that the triad λ, μ, ν is regular and that its polar triad is α, β, γ. The triad λ', μ', ν' is said to be the *Complementary-polar triad* of α, β, γ.

Corresponding to a regular triad of directed planes α, β, γ we thus get three related triads namely α', β', γ' the complementary triad, λ, μ, ν the polar triad, and λ', μ', ν' the complementary-polar triad. Starting with any of those four triads as the primary triad we find that the other three triads form the three related triads.

The two groups of directed-planes $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ and $\lambda, \mu, \nu, \lambda', \mu', \nu'$ are so related that any member of the first group is simply perpendicular to four members of the second group and vice versa. The relations connecting the mutual inclinations of these twelve

directed-planes are interesting. Out of the 132 angles between the ${}^{12}C_2$ pairs, 24 angles are 0 and an equal number $\pi/2$. The remaining 84 angles can be at once expressed in terms of the 18 angles between the nine pairs

$$\alpha, \beta;\ \beta, \gamma;\ \gamma, \alpha$$

$$\lambda, \mu;\ \mu, \nu;\ \nu, \lambda$$

$$\alpha, \lambda;\ \beta, \mu;\ \gamma, \nu.$$

by observing that if θ_1, θ'_1, be the angles between β and γ, then the the angles between β' and γ' are also θ_1 and θ'_1 while the angles between β' and γ or β and γ' are $\frac{\pi}{2}-\theta_1$ and $\frac{\pi}{2}-\theta'_1$. Finally, we show in the next article that only six of the 18 angles last mentioned are independent and in fact we can express each of them in terms of the six angles between α and β, β and γ, γ and α. Thus knowing the six angles between three directed-planes forming a regular triad, all the angles between the 12 directed planes, forming the given and the related triads, can be determined.

16. *Formulæ connecting the inclinations between directed planes forming a regular triad and its polar.*

Let α, β, γ be three directed-planes, with direction constants $a_i, b_i, c_i, f_i, g_i, h_i$, ($i=1, 2, 3$) respectively. Let λ, μ, ν be the polar triad of α, β, γ.

Let the angles between

$$\beta, \gamma;\ \gamma, \alpha;\ \alpha, \beta;$$

$$\mu, \nu;\ \nu, \lambda;\ \lambda, \mu;$$

$$\alpha, \lambda;\ \beta, \mu;\ \gamma, \nu;$$

be respectively

$$\theta_1, \theta'_1;\ \theta_2, \theta'_2;\ \theta_3, \theta'_3;$$

$$\phi_1, \phi'_1;\ \phi_2, \phi'_2;\ \phi_3, \phi'_3;$$

$$p_1, p'_1;\ p_2, p'_2;\ p_3, p'_3;$$

Let $$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $$\Delta=\begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix}$$

Let A_1, B_1 etc. be the prepared minors of a_1, b_1 etc. in D and F_1, G_1 etc. the prepared minors of f_1, g_1 etc. in Δ.

Then the direction constants of λ, μ, ν are

$$A_i/\sin(\theta_i-\theta'_i),\ B_i/\sin(\theta_i-\theta'_i),\ C_i/\sin(\theta_i-\theta'_i)$$

$$F_i/\sin(\theta_i+\theta'_i),\ G_i/\sin(\theta_i+\theta'_i),\ H_i/\sin(\theta_i+\theta'_i)$$

where $i=1, 2, 3$ respectively.

Hence by (37)

$$\cos(\phi_1-\phi'_1)=\frac{A_2A_3+B_2B_3+C_2C_3}{\sin(\theta_2-\theta'_2)\sin(\theta_3-\theta'_3)}$$

But $A_2A_3+B_2B_3+C_2C_3=(b_3c_1-b_1c_3)(b_1c_2-b_2c_1)$

$$+(c_3a_1-c_1a_3)(c_1a_2-c_2a_1)$$

$$+(a_3b_1-a_1b_3)(a_1b_2-a_2b_1)$$

$$=-(a_1^2+b_1^2+c_1^2)(a_2a_3+b_2b_3+c_2c_3)$$

$$+(a_3a_1+b_3b_1+c_3c_1)(a_1a_2+b_1b_2+c_1c_2)$$

$$=-\cos(\theta_1-\theta'_1)+\cos(\theta_2-\theta'_2)\cos(\theta_3-\theta'_3)$$

$$\left.\begin{array}{l}\therefore\ \cos(\phi_1-\phi'_1)=\dfrac{-\cos(\theta_1-\theta'_1)+\cos(\theta_2-\theta'_2)\cos(\theta_3-\theta'_3)}{\sin(\theta_2-\theta'_2)\sin(\theta_3-\theta'_3)}\\[2ex] \text{with similar expressions for } \cos(\phi_2-\phi'_2) \text{ and } \cos(\phi_3-\phi'_3)\end{array}\right\}\quad(62)$$

In the same way we can show that

$$\left.\begin{array}{l}\cos(\phi_1+\phi'_1)=\dfrac{-\cos(\theta_1+\theta'_1)+\cos(\theta_2+\theta'_2)\cos(\theta_3+\theta'_3)}{\sin(\theta_2+\theta'_2)\sin(\theta_3+\theta'_3)}\\[2ex] \text{with similar expressions for } \cos(\phi_2+\phi'_2) \text{ and } \cos(\phi_3+\phi'_3).\end{array}\right\}\quad(63)$$

Again $D^2=\begin{vmatrix} 1 & \cos(\theta_3-\theta'_3) & \cos(\theta_2-\theta'_2)\\ \cos(\theta_3-\theta'_3) & 1 & \cos(\theta_1-\theta'_1)\\ \cos(\theta_2-\theta'_2) & \cos(\theta_1-\theta'_1) & 1\end{vmatrix}$

$$=1-\cos^2(\theta_1-\theta'_1)-\cos^2(\theta_2-\theta'_2)-\cos^2(\theta_3-\theta'_3)$$

$$+2\cos(\theta_1-\theta'_1)\cos(\theta_2-\theta'_2)\cos(\theta_3-\theta'_3)\quad\ldots\quad(64)$$

Similarly $\Delta^2=1-\cos^2(\theta_1+\theta'_1)-\cos^2(\theta_2+\theta'_2)-\cos^2(\theta_3+\theta'_3)$

$$+2\cos(\theta_1+\theta'_1)\cos(\theta_2+\theta'_2)\cos(\theta_3+\theta'_3)\quad\ldots\quad(65)$$

Also,

$$\cos(p_1-p'_1)=\frac{a_1A_1+b_1B_1+c_1C_1}{\sin(\theta_1-\theta'_1)}$$

$$\therefore\ \cos(p_1-p'_1)\cos(\theta_1-\theta'_1)=\cos(p_2-p'_2)\sin(\theta_2-\theta'_2)$$

$$=\cos(p_3-p'_3)\sin(\theta_3-\theta'_3)=D\quad\ldots\quad(66)$$

Similarly

$$\cos(p_1+p'_1)\cos(\theta_1+\theta'_1)=\cos(p_2+p'_2)\sin(\theta_2+\theta'_2)$$

$$=\cos(p_3+p'_3)\sin(\theta_3+\theta'_3)=\Delta\quad\ldots\quad(67)$$

Now as α, β, γ is the polar triad of λ, μ, ν we can interchange the θ's and ϕ's in (62), (63), (66) and (67).

Thus

$$\left.\begin{array}{l}\cos(\theta_1-\theta'_1)=\dfrac{-\cos(\phi_1-\phi'_1)+\cos(\phi_2-\phi'_2)\cos(\phi_3-\phi'_3)}{\sin(\phi_2-\phi'_2)\sin(\phi_3-\phi'_3)} \\ \text{with similar expressions for } \cos(\theta_2-\theta'_2) \text{ and } \cos(\theta_3-\theta'_3)\end{array}\right\} \quad (68)$$

$$\left.\begin{array}{l}\cos(\theta_1+\theta'_1)=\dfrac{-\cos(\phi_1+\phi'_1)+\cos(\phi_2+\phi'_2)\cos(\phi_3+\phi'_3)}{\sin(\phi_2+\phi'_2)\sin(\phi_3+\phi'_3)} \\ \text{with similar expressions for } \cos(\theta_2+\theta'_2) \text{ and } \cos(\theta_3+\theta'_3)\end{array}\right\} \quad (69)$$

$$\cos(p_1-p'_1)\sin(\phi_1-\phi'_1)=\cos(p_2-p'_2)\sin(\phi_2-\phi'_2)$$

$$=\cos(p_3-p'_3)\sin(\phi_3-\phi'_3) \quad \ldots \quad (70)$$

$$\cos(p_1+p'_1)\sin(\phi_1+\phi'_1)=\cos(p_2+p'_2)\sin(\phi_2+\phi'_2)$$

$$=\cos(p_3+p'_3)\sin(\phi_3+\phi'_3) \quad \ldots \quad (71)$$

Also from (66), (67), (70) and (71)

$$\frac{\sin(\theta_1-\theta'_1)}{\sin(\phi_1-\phi'_1)}=\frac{\sin(\theta_2-\theta_2')}{\sin(\phi_2-\phi'_2)}=\frac{\sin(\theta_3-\theta'_3)}{\sin(\phi_3-\phi'_3)} \quad \ldots \quad (72)$$

$$\frac{\sin(\theta_1+\theta'_1)}{\sin(\phi_1+\phi'_1)}=\frac{\sin(\theta_2+\theta'_2)}{\sin(\phi_2+\phi'_2)}=\frac{\sin(\theta_3+\theta'_3)}{\sin(\phi_3+\phi'_3)} \quad \ldots \quad (73)$$

The similarity between the formulæ (62) to (73) with the formulæ connecting the sides and angles of a spherical triangle must have been noticed. The followlng consideratious bring out the exact relationship between the two sets of formulæ.

Let

$$\theta_1-\theta'_1=p, \qquad \theta_2-\theta'_2=q, \qquad \theta_3-\theta'_3=r$$

$$\phi_1-\phi'_1=\pi-P, \qquad \phi_2-\phi'_2=\pi-Q, \qquad \phi_3-\phi'_3=\pi-R$$

The inequalities (28) show that each of p, q, r, P, Q, R lies between 0 and π.

Again from (62)

$$\left.\begin{aligned}\cos p&=\cos q\cos r+\sin q\sin r\cos P\\ \cos q&=\cos r\cos p+\sin r\sin p\cos Q\\ \cos r&=\cos p\cos q+\sin p\sin q\cos R\end{aligned}\right\}$$

p, q, r, P, Q, R are thus the elements of a spherical triangle.

Also

$$\cos(p_i-p'_i)=\sqrt{1-\cos^2 p-\cos^2 q-\cos^2 r+2\cos p\cos q\cos r}$$

$$(i=1, 2, 3)$$

It is thus evident that

If $\theta_1-\theta'_1$, $\theta_2-\theta'_2$, $\theta_3-\theta'_3$ be the sides of a spherical triangle, then $\phi_1-\phi'_1$, $\phi_2-\phi'_2$, $\phi_3-\phi'_3$ are the corresponding sides of the polar triangle and $p_1-p'_1$, $p_2-p'_2$, $p_3-p'_3$ are the joins of the corresponding vertices of the two triangles.

Similarly

If $\theta_1+\theta'_1$, $\theta_2+\theta'_2$, $\theta_3+\theta'_3$ be the sides of a spherical triangle then $\phi_1+\phi'_1$, $\phi_2+\phi'_2$, $\phi_3+\phi'_3$ are the sides of the polar triangle and $p_1+p'_1$, $p_2+p'_2$, $p_3+p'_3$ are the joins of the corresponding vertices of the two triangles.

Thus corresponding to any relation between the elements of a spherical triangle we can at once write down two relations between the mutual inclinatiens of the directed planes λ, μ, ν, λ', μ', ν'.

17. *Relations between the mutual inclinations of four arbitrary directed planes.*

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be four directed-planes and let $\theta_{ij}, \theta'_{ij}$ be the angles between α_i and α_j $(i=1, 2, 3, 4, \quad j=i+1, \quad \ldots 4)$

Let $a_i, b_i, c_i, f_i, g_i, h_i$ be the direction constants of α_i $(i=1, 2, 3, 4)$.

We then have

$$\left.\begin{aligned} \cos(\theta_{ij}-\theta'_{ij}) &= a_i a_j + b_i b_j + c_i c_j \\ \cos(\theta_{ij}+\theta'_{ij}) &= f_i f_j + g_i g_j + h_i h_j \end{aligned}\right\}$$

where $i=1, 2, 3, 4, \quad j=i+1, \quad \ldots \quad 4.$

The determinant obtained by squaring the rectangular array

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}$$

must identically vanish.

Hence

$$\begin{vmatrix} 1 & \cos(\theta_{12}-\theta'_{12}) & \cos(\theta_{13}-\theta'_{13}) & \cos(\theta_{14}-\theta'_{14}) \\ \cos(\theta_{12}-\theta'_{12}) & 1 & \cos(\theta_{23}-\theta'_{23}) & \cos(\theta_{24}-\theta'_{24}) \\ \cos(\theta_{13}-\theta'_{13}) & \cos(\theta_{23}-\theta'_{23}) & 1 & \cos(\theta_{34}-\theta'_{34}) \\ \cos(\theta_{14}-\theta'_{14}) & \cos(\theta_{24}-\theta'_{24}) & \cos(\theta_{34}-\theta'_{34}) & 1 \end{vmatrix} = 0 \quad (74)$$

Similarly

$$\begin{vmatrix} 1 & \cos(\theta_{12}+\theta'_{12}) & \cos(\theta_{13}+\theta'_{13}) & \cos(\theta_{14}+\theta'_{14}) \\ \cos(\theta_{12}+\theta'_{12}) & 1 & \cos(\theta_{23}+\theta'_{23}) & \cos(\theta_{24}+\theta'_{24}) \\ \cos(\theta_{13}+\theta'_{13}) & \cos(\theta_{23}+\theta'_{23}) & 1 & \cos(\theta_{34}+\theta'_{34}) \\ \cos(\theta_{14}+\theta'_{14}) & \cos(\theta_{24}+\theta'_{24}) & \cos(\theta_{34}+\theta'_{34}) & 1 \end{vmatrix} = 0 \quad (75)$$

In conclusion my best thanks are due to Dr. S Mukhopadhyya under whose guidance I carried on the investigation the results of which are embodied in this paper.

6

On a type of solution of Einstein's gravitational equations

(Part II)

BY

Nripendra Nath Ghosh

1. In the first part of this paper, recently published in the *Bulletin*,* it is shewn that if the line element be of the type

$$ds^2 = a_1 e^{2h_{11}} dx_1^2 + a_2 e^{2h_{22}} dx_2^2 + a_3 e^{2h_{33}} dx_3^2 + a_4 e^{2h_{44}} dx_4^2$$

where h's are functions of x_1, x_2, x_3, x_4 and a's are numerical constants (introduced to manipulate the signs), Einstein's gravitational equations

$$G_{\mu\nu} = 0, \quad (\mu, \nu = 1, 2, 3, 4)$$

are reducible to the forms

$$\text{(A)} \quad G_{pp} \equiv \sum_{\lambda}^{(m,n,r)} \Bigg[\binom{p}{\lambda} e^{2(h_{pp}-h_{\lambda\lambda})} \left\{ \frac{\partial^2}{\partial x_\lambda^2} h_{pp} + \frac{\partial}{\partial x_\lambda} h_{pp} \frac{\partial}{\partial x_\lambda} (h - 2h_{\lambda\lambda}) \right\}$$

$$+ \frac{\partial^2}{\partial x_p^2} h_{\lambda\lambda} + \left(\frac{\partial}{\partial x_p} h_{\lambda\lambda} \right)^2 - \frac{\partial}{\partial x_p} h_{pp} \frac{\partial}{\partial x_p} h_{\lambda\lambda} \Bigg] = 0,$$

* *Bull. Cal. Math. Soc.*, Vol. XVII, No. 1.

$$(B) \quad G_{mn} \equiv \sum_{\lambda}^{(r,p)} \left[\frac{\partial^2}{\partial x_m \partial x_n} h_{\lambda\lambda} + \frac{\partial}{\partial x_m} h_{\lambda\lambda} \frac{\partial}{\partial x_n} h_{\lambda\lambda} \right.$$

$$\left. - \frac{\partial}{\partial x_n} h_{mm} \frac{\partial}{\partial x_m} h_{\lambda\lambda} - \frac{\partial}{\partial x_m} h_{nn} \frac{\partial}{\partial x_n} h_{\lambda\lambda} \right] = 0,$$

where m, n, r, p represent the four numbers 1, 2, 3, 4 taken in any order, $\left(\begin{smallmatrix} p \\ \lambda \end{smallmatrix} \right)$ denotes the ratio $\frac{a_p}{a_\lambda}$ and h stands for the expression

$$h_{mm} + h_{nn} + h_{rr} + h_{pp}.$$

By direct integration of the above equations some particular solutions of Einstein's gravitational equations may be obtained, one of which due to Schwarzschild has been discussed in the first part. The aim of the present paper is to study two more cases, one of them leading to a solution believed to be new.

The cases to be considered are those in which

(II) h_{11} and h_{44} are functions of x_1 only, while h_{22} and h_{33} are each a function of x_1, x_2;

(III) h_{11} is a function of x_1 only, while each of h_{22}, h_{33}, h_{44} is a function of x_1, x_2.

2. We shall take up case (III) first. Choosing

$$a_1 = a_2 = a_3 = -a_4 = -1,$$

we have the following five differential equations to be dealt with :

$$G_{11} \equiv \frac{\partial^2}{\partial x_1^2} (h_{22} + h_{33} + h_{44}) + \left(\frac{\partial}{\partial x_1} h_{22} \right)^2$$

$$+ \left(\frac{\partial}{\partial x_1} h_{33} \right)^2 + \left(\frac{\partial}{\partial x_1} h_{44} \right)^2$$

$$- \frac{\partial}{\partial x_1} h_{11} \frac{\partial}{\partial x_1} (h_{22} + h_{33} + h_{44}) = 0,$$

$$G_{22} \equiv e^{2(h_{22}-h_{11})} \left\{ \frac{\partial^2}{\partial x_1^2} h_{22} \right.$$

$$\left. + \frac{\partial}{\partial x_1} h_{22} \frac{\partial}{\partial x_1} (h_{22}+h_{33}+h_{44}-h_{11}) \right\} + \frac{\partial^2}{\partial x_2^2} (h_{33}+h_{44})$$

$$+ \left(\frac{\partial}{\partial x_2} h_{33} \right)^2 + \left(\frac{\partial}{\partial x_2} h_{44} \right)^2 - \frac{\partial}{\partial x_2} h_{22} \frac{\partial}{\partial x_2} (h_{33}+h_{44}) = 0,$$

$$G_{33} \equiv e^{2(h_{33}-h_{11})} \left\{ \frac{\partial^2}{\partial x_1^2} h_{33} \right.$$

$$\left. + \frac{\partial}{\partial x_1} h_{33} \frac{\partial}{\partial x_1} (h_{22}+h_{33}+h_{44}-h_{11}) \right\}$$

$$+ e^{2(h_{33}-h_{22})} \left\{ \frac{\partial^2}{\partial x_2^2} h_{33} \right.$$

$$\left. + \frac{\partial}{\partial x_2} h_{33} \frac{\partial}{\partial x_2} (h_{33}+h_{44}-h_{22}) \right\} = 0,$$

$$G_{44} \equiv -e^{2(h_{44}-h_{11})} \left\{ \frac{\partial^2}{\partial x_1^2} h_{44} \right.$$

$$\left. + \frac{\partial}{\partial x_1} h_{44} \frac{\partial}{\partial x_1} (h_{22}+h_{33}+h_{44}-h_{11}) \right\}$$

$$- e^{2(h_{44}-h_{22})} \left\{ \frac{\partial^2}{\partial x_2^2} h_{44} \right.$$

$$\left. + \frac{\partial}{\partial x_2} h_{44} \frac{\partial}{\partial x_2} (h_{33}+h_{44}-h_{22}) \right\} = 0,$$

$$G_{12} \equiv \frac{\partial^2}{\partial x_1 \partial x_2} (h_{33}+h_{44}) + \frac{\partial}{\partial x_2} h_{33} \frac{\partial}{\partial x_1} h_{33} + \frac{\partial}{\partial x_2} h_{44} \frac{\partial}{\partial x_1} h_{44}$$

$$- \frac{\partial}{\partial x_1} h_{22} \frac{\partial}{\partial x_2} (h_{33}+h_{44}) = 0.$$

The above equations are considerably simplified if we start with the assumption

$$\frac{\partial}{\partial x_2}(h_{33}+h_{44})=0. \qquad \dots \quad (1)$$

From G_{12} we have then

$$\frac{\partial}{\partial x_1}h_{33}=\frac{\partial}{\partial x_1}h_{44}. \qquad \dots \quad (2)$$

The equations G_{33} and G_{44} are now equivalent to

$$\frac{\partial^2}{\partial x_1^2}h_{33}+\frac{\partial}{\partial x_1}h_{33}\frac{\partial}{\partial x_1}(h_{22}+2h_{33}-h_{11})=0, \qquad \dots \quad (3)$$

$$\frac{\partial^2}{\partial x_2^2}h_{33}-\frac{\partial}{\partial x_2}h_{33}\frac{\partial}{\partial x_2}h_{22}=0. \qquad \dots \quad (4)$$

The equations G_{11} and G_{22} reduce respectively to

$$\frac{\partial^2}{\partial x_1^2}h_{22}+\left(\frac{\partial}{\partial x_1}h_{22}\right)^2-\frac{\partial}{\partial x_1}h_{11}\frac{\partial}{\partial x_1}h_{22}$$

$$=2\frac{\partial}{\partial x_1}h_{33}\frac{\partial}{\partial x_1}(h_{33}+h_{22}) \qquad \dots \quad (5)$$

and $$\left(\frac{\partial}{\partial x_2}h_{33}\right)^2+e^{2(h_{22}-h_{11})}\left\{\left(\frac{\partial}{\partial x_1}h_{33}\right)^2\right.$$

$$\left.+2\frac{\partial}{\partial x_1}h_{33}\frac{\partial}{\partial x_1}h_{22}\right\}=0 \qquad \dots \quad (6)$$

Putting $$\frac{\partial}{\partial x_1} h_{22} = -N \frac{\partial}{\partial x_1} h_{33}.$$

and using the symbols

$$H_{\mu\mu} \quad \text{and} \quad K_{\mu\mu} \quad \text{for} \quad \frac{\partial}{\partial x_1} h_{\mu\mu} \quad \text{and} \quad \frac{\partial}{\partial x_2} h_{\mu\mu}$$

respectively, we can replace equations (3), (4), (5), (6) by means of

$$\frac{\partial}{\partial x_1} H_{33} + (2-N) H_{33}^2 - H_{33} H_{11} = 0, \qquad \ldots \quad (7)$$

$$\frac{\partial}{\partial x_2} K_{33} - K_{33} K_{22} = 0, \qquad \ldots \quad (8)$$

$$(4N-2) H_{33} = \frac{\partial N}{\partial x_1}, \qquad \ldots \quad (9)$$

$$K_{33}^2 = e^{2(h_{22}-h_{11})} \{(2N-1) H_{33}^2\}. \qquad \ldots \quad (10)$$

It can now be shown that

$$\frac{\partial}{\partial x_1} \log e^{2(h_{22}-h_{11})} \{(2N-1) H_{33}^2\} = 0;$$

so that from (10) we have

$$\frac{\partial}{\partial x_1} \log K_{33} = 0, \qquad i.e., \quad \frac{\partial^2}{\partial x_1 \partial x_2} h_{33} = 0.$$

Therefore h_{33} must be of the form

$$\log \phi_1(x_1) \phi_2(x_2). \qquad \ldots \quad (11)$$

Consequently, h_{44} will be

$$\log \frac{\phi_1(x_1)}{\phi_2(x_2)} c_4 \quad \text{where } c_4 \text{ is an arbitrary constant.} \quad \dots \quad (12)$$

Equation (7) gives

$$H_{11} = \frac{\phi_1''(x_1)}{\phi_1'(x_1)} + (1-N)\frac{\phi_1'(x_1)}{\phi_1(x_1)},$$

whence h_{11} is obtained.

Since H_{11} is a function of x_1 only we infer from the above that N must be a function of x_1 also. By integrating equation (9) its value is found to be

$$\tfrac{1}{2} + \tfrac{c}{2}\{\phi_1(x_1)\}^4 \quad \text{where } c \text{ is a constant of integration.}$$

It follows immediately that

$$h_{11} = \log \phi_1'(x_1)c_1 + \tfrac{1}{2}\log \phi_1(x_1) - \tfrac{c}{8}\{\phi_1(x_1)\}^4, \quad \dots \quad (13)$$

$$h_{22} = -\tfrac{1}{2}\log \phi_1(x_1) - \tfrac{c}{8}\{\phi_1(x_1)\}^4 + \log \phi_2'(x_2)c_2 - \log \phi_2(x_2), \quad \dots \quad (14)$$

where c_1 and c_2 are arbitrary constants.

Thus we have

$$e^{2h_{11}} = \{\phi_1'(x_1)\}^2 \phi_1(x_1) c_1^2 e^{-\frac{c}{4}\{\phi_1(x_1)\}^4},$$

$$e^{2h_{22}} = \{\phi_2'(x_2)\}^2 \frac{1}{\{\phi_2(x_2)\}^2 \phi_1(x_1)} \cdot c_2^2 e^{-\frac{c}{4}\{\phi_1(x_1)\}^4},$$

$$e^{2h_{33}} = \{\phi_1(x_1)\phi_2(x_2)\}^2,$$

$$e^{2h_{44}} = \{\phi_1(x_1)\}^2 \times \frac{1}{\{\phi_2(x_2)\}^2} c_4^2,$$

where the arbitrary constants c, c_1^2, c_2^2 satisfy the condition $cc_2^2 = c_1^2$ (found by substitution in (10).

Hence the required line element reduces after slight transformations to:

$$ds^2 = -\alpha\beta x_1 e^{-\frac{\alpha}{4}x_1^4}\, dx_1^2 - \frac{\beta}{x_1 x_2^2} e^{-\frac{\alpha}{4}x_1^4}\, dx_2^2$$

$$-x_1^2 x_2^2 dx_3^2 + \frac{x_1^2}{x_2^2}\, dx_4^2, \qquad \ldots \quad (15)^*$$

where α and β are two arbitrary constants.

3. The geodesic curves are given by

$$\frac{d^2 x_\alpha}{ds^2} + \{\mu\nu, \alpha\} \frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = 0 \quad (\mu, \nu, \alpha = 1, 2, 3, 4).$$

In the space-time manifold given by (15), these equations appear in the form:

$$\frac{dx_3}{ds} = \frac{A}{x_1^2 x_2^2}, \qquad \ldots \quad (i)$$

$$\frac{dx_4}{ds} = \frac{Bx_2^2}{x_1^2}, \qquad \ldots \quad (ii)$$

$$\frac{d^2 x_1}{ds^2} + \tfrac{1}{2}\left(\frac{1}{x_1} - \alpha x_1^3\right)\left(\frac{dx_1}{ds}\right)^2 + \frac{1+\alpha x_1^4}{2\alpha x_1^3 x_2^2}\left(\frac{dx_2}{ds}\right)^2$$

$$+ \frac{e^{\frac{\alpha}{4}x_1^4}}{\alpha\beta x_1^4}\left(B^2 x_2^2 - \frac{A^2}{x_2^2}\right) = 0, \qquad \ldots \quad (iii)$$

$$\frac{d^2 x_2}{ds^2} - \frac{1}{x_2}\left(\frac{dx_2}{ds}\right)^2 - \left(\frac{1}{x_1} + \alpha x_1^3\right)\frac{dx_1}{ds}\frac{dx_2}{ds}$$

$$- \frac{e^{\frac{\alpha}{4}x_1^4}}{\beta x_1}\left(\frac{A^2}{x_2} + B^2 x_2^3\right) = 0, \qquad \ldots \quad (iv)$$

where A and B are arbitrary constants.

* It may be noted that the co-efficient of dx_2^2 may be reduced to a function of x_1 only by the substitution $\log x_2 = \xi$. The two subsequent terms require then corresponding modifications.

The above equations are complicated and do not admit of easy solution. We shall, however, only examine if there is any geodesic curve on certain "planes."

We, therefore, put

$$\frac{dx_1}{ds}=0, \quad \frac{d^2x_1}{ds^2}=0$$

in the last two equations and obtain

$$\frac{1}{2}\left(\frac{1}{\alpha x_1^3}+x_1\right)\left(\frac{dx_2}{ds}\right)^2=\frac{e^{\frac{\alpha}{4}x_1^4}}{\alpha\beta x_1^4}\left(A^2-B^2x_2^4\right) \qquad \dots \quad (v)$$

$$\frac{d^2x_2}{ds^2}-\frac{1}{x_2}\left(\frac{dx_2}{ds}\right)^2=-\frac{e^{\frac{\alpha}{4}x_1^4}}{\beta x_1}\left(\frac{A^2}{x_2}+B^2x_2^3\right) \qquad \dots \quad (vi)$$

The equations (v) and (vi) are consistent if

$$x_1^4=-\frac{3}{\alpha}, \qquad \dots \quad (vii)$$

so that there are geodesic curves on the "planes" given by (vii).

4. Let us now proceed with case (II). Choosing α's as before the differential equations may be written down by putting $\frac{\partial}{\partial x_1}h_{44}=0$ in the set of 5 equations for case (III). The equations can be tackled as they are.

From G_{22} and G_{33} we get

$$\frac{\partial^2}{\partial x_1^2}(h_{22}-h_{33})+\left(\frac{\partial}{\partial x_1}h_{22}\right)^2-\left(\frac{\partial}{\partial x_1}h_{33}\right)^2$$

$$+\frac{\partial}{\partial x_1}(h_{44}-h_{11})\frac{\partial}{\partial x_1}(h_{22}-h_{33})=0, \qquad \dots \quad (1)$$

and from G_{11} and G_{44} we have

$$\frac{\partial^2}{\partial x_1^2}(h_{22}+h_{33})+\left(\frac{\partial}{\partial x_1}h_{22}\right)^2+\left(\frac{\partial}{\partial x_1}h_{33}\right)^2$$

$$-\frac{\partial}{\partial x_1}(h_{22}+h_{33})\frac{\partial}{\partial x_1}(h_{44}+h_{11})=0. \qquad \dots \quad (2)$$

Equations (1) and (2) are equivalent to

$$\frac{\partial^2}{\partial x_1^2} h_{22} + \left(\frac{\partial}{\partial x_1} h_{22} \right)^2 = \frac{\partial}{\partial x_1} h_{11} \frac{\partial}{\partial x_1} h_{22}$$

$$+ \frac{\partial}{\partial x_1} h_{33} \frac{\partial}{\partial x_1} h_{44}, \quad \ldots \quad (3)$$

$$\frac{\partial^2}{\partial x_1^2} h_{33} + \left(\frac{\partial}{\partial x_1} h_{33} \right)^2 = \frac{\partial}{\partial x_1} h_{11} \frac{\partial}{\partial x_1} h_{33}$$

$$+ \frac{\partial}{\partial x_1} h_{22} \frac{\partial}{\partial x_1} h_{44}. \quad \ldots \quad (4)$$

The equation G_{44} indicates that $\frac{\partial}{\partial x_1}(h_{22}+h_{33})$ cannot involve x_2 and from equation (2) it further follows that $\left(\frac{\partial}{\partial x_1} h_{22} \right)^2 + \left(\frac{\partial}{\partial x_1} h_{33} \right)^2$ cannot involve x_2, whence we may infer that each of $\frac{\partial}{\partial x_1} h_{22}$ and $\frac{\partial}{\partial x_1} h_{33}$ does not involve x_2 and the set of equations is then consistent.

By proceeding almost exactly in the manner indicated in the first part of this paper it is not difficult to obtain h's as follows:

$$h_{11} = \log \left(\frac{2+M}{M} \right)^{\frac{1}{2}} \frac{\partial M}{\partial x_1} c_1,$$

$$h_{22} = \log (2+M) c_1 \phi_2,$$

$$h_{33} = \log (2+M) \phi_3,$$

$$h_{44} = \tfrac{1}{2} \log \frac{M}{2+M} c_4,$$

where c_1 and c_4 are arbitrary constants and ϕ_2, ϕ_3 (two functions of x_2) satisfy the equation

$$\phi_3''\phi_2 - \phi_3'\phi_2' + \phi_3\phi_2^3 = 0.$$

The most general solution of the above is given by

$$\phi_3 = b_1 \sin \{\int \phi_2 dx_2 + b_2\}$$

where b_1 and b_2 are arbitrary constants.

The required line element is therefore

$$ds^2 = -\frac{2+M}{M} c_1^2 dM^2 - (2+M)^2 c_1^2 \phi_2^2 dx_2^2$$

$$-(2+M)^2 b_1^2 \sin^2 \{\int \phi_2 dx_2 + b_2\} + \frac{M}{2+M} c_4 dx_4^2$$

which ultimately reduces by suitable transformations to Schwarzschild's form

$$ds^2 = -\frac{1}{1-\frac{2c_1}{x_1}} dx_1^2 - x_1^2 dx_2^2 - x_1^2 \sin^2 x_2 dx_3^2 + \left(1 - \frac{2c_1}{x_1}\right) dx_4^2.$$

Hence Einstein's equations admit of no new solution of type (II), *any solution of this nature being reducible to Schwarzschild's form.*

In conclusion, I wish to express my thanks to Dr. N. R. Sen for constant guidance in course of this investigation.

CORRIGENDA.

(*Reference—Bull. Cal. Math. Soc., Vol. XVII, No.* 1)

Page 40,	line 19,	*for* Z	*read* z
„ 40,	„ 20,	„ $\because$	„ $\therefore$
„ 40,	„ 25,	„ $\because$	„ $\therefore$
„ 42,	„ 7,	„ P	„ p
„ 48,	„ 2,	„ from	„ from definition
„ 48,	„ 17,	„ H	„ H′
„ 48,	„ 29,	„ between to	„ between two
„ 53,	„ 3,	„ meet	„ meet at

1

GENESIS OF AN ELEMENTARY ARC

BY

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1. *Introductory.*

The development of the *theory* of *elementary curves* is primarily due to *C. Juel* of Copenhagen. *P. Montel* has reviewed C. Juel's work in *the Bulletin des Sciences Mathematiques*, 1924, part I, as also that of S. Mukhopadhyaya on similar lines which have a certain priority of origin. A bibliography on the subject occurs at the end of P. Montel's review.

C. Juel's concept of an elementary arc is exposed by P. Montel as follows :—

"It is necessary above all, to define the simple element which serves as the basis for the construction of plane (elementary) curves, which we proceed in the first place to study with M. Juel. Let us imagine an arc of a continuous curve with extremities A and B ; if this arc encloses with the chord AB, a convex domain, one can easily deduce from this the existence at each point of the arc an anterior half-tangent and a posterior half-tangent. To this let us add the condition that these half-tangents have the same direction ; our arc shall then possess, at each point, a tangent varying in a continuous manner with the point of contact. We shall thus obtain an *elementary arc*. Such an arc is met in two points at most by a straight line ; one can draw to it two tangents at most from a point."

The definition of an elementary arc as outlined above, assumes that we know how to define a continuous curve in a satisfactory way—a thing which we perhaps do not know. The arc has undefined proportions and as such is of more limited use than the one defined in this paper.

The way in which an elementary arc has been evolved in this paper from a chain of cellular elements may prove interesting to geometers as a novel solution of the problem of the plane elementary arc on rigorous lines.

2. Consider an ordered set of a finite number of points A, P_1, P_2, ...P_{n-1}, B in a restricted domain on a plane which may be Euclidean

or non-Euclidean. The train of n sects AP_1, $P_1P_2, \ldots\ldots P_{n-1}B$ constitutes a *linear chain* of *rank* n. The points $A, P_1, P_2, \ldots\ldots P_{n-1}$, B will be supposed all distinct, except that B may coincide with A. In the latter case the chain is *closed* and in the former case the chain is *open*.

In the open linear chain of rank n there are $n-1$ vertices $P_1, P_2, \ldots, P_{n-1}$ and two extremities A and B. In the closed linear chain of rank n there are n vertices and no extremities. The order $A, P_1, P_2, \ldots\ldots P_{n-1}$, B will be called the *positive* order on the chain as distinguished from the order $B, P_{n-1}, \ldots\ldots P_2, P_1$, A which will be called the *negative* order on the chain.

Each of the sects AP_1, $P_1P_2, \ldots\ldots P_{n-1}B$ will be called a *trace* of the chain. The trace PQ will be considered positive or negative according as P precedes or succeeds Q in the positive order on the chain. The extremities P and Q will be included in the trace PQ. Two consecutive traces PQ, QR can have only one point Q common unless they overlap. If no two non-consecutive traces have a common point and if two consecutive traces have only one point common, the chain will be called *simple*.

3. If PQ and QR be any two consecutive traces of a simple chain, P, Q, R being in positive order, then QR will be either to the right or to the left of PQ or in the prolongation of PQ. In the first case the chain will be said to have a *positive trend*, in the second a *negative trend* and in the third case a *zero trend*, at the vertex Q. The absolute amount of the trend at Q is measured by an angle less than two right angles between the directions of PQ and QR taken positively.

If a simple chain has at every vertex Q a trend of the same sign, with the possibility of a zero trend at some, the chain will be called *monocline* or of *unilateral trend*. A monocline chain may be either positively or negatively so, that is, it may be either of *dextrolateral* or of *levo-lateral* trend.

A simple closed mono-cline chain is called a *convex polygon*. We may suppose that in a convex polygon the trend does not vanish at any vertex, so that there are exactly *n bounding lines* in a convex polygon of rank n, consisting of the n traces of the simple closed mono-cline chain which defines it.

Theorems.

4. (*i*) *A convex polygon lies entirely on the same side of each of its bounding lines, that is if PQ be any bounding line, taken in the positive sense, all the other bounding lines will fall on the right side or left side of PQ according as the polygon is positively or negatively monocline, respectively.*

(*ii*) *No straight line which does not pass through two consecutive vertices can meet a convex polygon at more than two distinct points.*

It is usual to assume *Theorem* (*i*) as the distinguishing property of a convex polygon and to deduce *Theorem* (*ii*) from it. Theorem (*i*) however can be proved from definition of a convex polygon as follows :

Suppose if possible, that such a polygon lies partly on one side and partly on the other side of a bounding line PQ. Suppose NP and QR are respectively the bounding lines which immediately precede and succeed PQ, N, P, Q, R being in positive order on the polygon, which we will suppose, has a dextro-lateral trend. Then NP and QR lie on the right-side of PQ, but as part of the polygon lies to the left of PQ by hypothesis, PQ meets the polygon again at some point X. Suppose X lies on PQ produced towards Q, so that the part of the polygon between Q and X lies wholly to the right of PQ, as QR is to the right of PQ. Turn QX about Q towards the right till QX falls along QR. Then X will either coincide with R or have a distinct position X′ on QR produced towards R. In the former case, suppose RS is the bounding line immediately succeeding QR, so that X finally travels along RS to reach R. RS is therefore to the left of QR whereas PQ is to the right of QR, which is impossible as the polygon has a unilateral trend.

In the latter case, turn RX′ again to the rignt till RX′ falls along RS. Then X′ will either coincide with S or will have a distinct position on RS produced towards S.

The former is impossible and the latter leads to the repetition of the process of rotation to the right. But the number of vertices of the polygon which may lie between R and X is finite and consequently the number of possible rotations to the right will soon be exhausted, rendering the alternative position of X impossible. Thus *Theorem* (*i*) cannot be false.

To prove *Theorem* (*ii*), suppose, if possible, a straight line other than a bounding line meets the polygon at three distinct points U, V, W, in positive order on the polygon. Then V must also lie between U and W on the straight line UW as the polygonal chain is simple. Suppose V is an interior point or end-point of the bounding line PQ, So that U and W lie on opposite sides of PQ. The portions UP and QW of the polygon will therefore lie wholly or partly on opposite sides of PQ. This contradicts *Theorem* (*i*).

5. Consider a simple chain A P_1 P_2 ...... P_{n-1} B of unilateral trend all of whose vertices lie on the same side of AB. Such a chain may be called a ***convex chain.***

Suppose all the vertices of a convex chain $AP_1P_2 \ldots\ldots P_{n-1}B$ are *interior* points of a triangle ATB ,such that the angle between AT produced and TB is less than a given acute angle α. Also suppose AB is less than a certain length l, so that the exterier angle theorem holds for the domain enclosed by the triangle. The triangle ATB will be called the *principal cell* of the chain and the chain $AP_1P_2 \ldots\ldots P_{n-1}B$ will be called *an elementary chain* of cell-angle$<\alpha$ and base $AB<l$.

If NP, PQ, QR be any three consecutive traces of an elementary chain in cell ATB then each of these traces produced positively will meet TB and produced negatively will meet AT. Consequently NP produced positively and QR produced negatively will intersect at some point U interior to the triangle ATB, such that the angle between the positive directions of NP and QR is less than α. Also $PQ<AB<l$, for if PQ produced meets AT at U and BT at W, then

$$PQ<VW<VB<AB.$$

The triangle PUQ will be called an *elementary cell* on trace PQ or carried by trace PQ of the elementary chain $A\ P_1P_2 \ldots\ldots\ldots\ldots P_{n-1}B$.

The elementary cell on initial trace AP will be a triangle AXP, where X is the intersection of P_1P_2 produced negatively and a line AX which lies between AT and AP and determined in any consistent manner. Similarly the elementary cell on final trace $P_{n-1}B$ is a triangle BYP_{n-1} where Y is the intersection of $P_{n-2}P_{n-1}$ produced positively and a line BY which lies between BP_{n-1} and BT and determined in any consistent manner.

The elementary cells carried by the successive traces of a given elementary linear chain form *an elementary cellular chain* carried by a given elementary linear chain. It may be observed that each elementry cell falls entirely inside the principal cell of the chain with the exception of the initial and final elementary cells which have a corner at A and B respectively. If PQ and RS are two non-adjacent traces of the elementary chain then the corresponding elementary cells will lie entirely outside one another.

6. The length of the longest trace of a linear chain will be called the *head* of the traces and that of the shortest trace will be called the *tail* of the traces. The magnitude of the largest of the elementary cell-angles of a cellular chain will be called the *head* of the cell angles and that of the smallest of the cell-angles will be called the *tail* of the cell-angles.

If the rank of a given elementary chain c be increased by the interpolations of additional vertices between pairs of consecutive vertices of the given chain and the new chain c' thus obtained be

also elementary, then c' will be called a *gemmatic extension* of c or gemmatically derived from c, *provided*

(*i*) the order of the vertices of c is the same in c and c'.

(*ii*) the extremities of c and c' are the same.

(*iii*) the principal cell AT′B of c' is the same as the principal cell of c or falls within it.

(*iv*) the initial and final elementary cells of c' fall within the initial and final elementary cell respectively of c with the points A and B respectively common.

If PQ be a trace of c and P′Q′ of c' such that the vertices P′, Q′ of c' fall between P, Q, *i.e.*, the points P, P′, Q′, Q are vertices of c' in order, then P′Q′ is said to have been gemmatically derived from PQ. P′ may however coincide with P or Q′ with Q. The elementary cell carried by P′Q′ in c' is also said to have been gemmatically derived from the elementary cell carried by PQ in c.

7. A system of elementary linear chains $c_1, c_2, \ldots\ldots\ldots, c_r, \ldots\ldots\ldots$ such that each chain except the first is gemmatically derived from the one just preceding it, will be called a *gemmatic system of elementary chains.*

Similarly a system of cellular chains carried by a gemmatic system of elementary linear chains will be called gemmatic.

Each of the above two systems will be called *regular* if the heads of the traces of $c_1, c_2, \ldots\ldots\ldots c_r, \ldots\ldots\ldots$ form a monotone decreasing sequence of zero limit and the heads of the elementary cell-angles of $c_1, c_2, \ldots c_r, \ldots$ also form a monotone sequence of zero limit.

A sequence of traces $t_1, t_2, \ldots\ldots\ldots, t_r, \ldots\ldots\ldots$ belonging respectively to chains $c_1, c_2, \ldots\ldots\ldots, c_r$, of a regular gemmatic system, which are such that each except the first is gemmatically derived from the one just preceding it, will be called a *regular gemmatic sequence of traces.* The corresponding elementary cells belonging to $c_1, c_2, \ldots\ldots, c_r, \ldots\ldots\ldots$ respectively will be called a *regular gemmatic sequence of cells.* A regular gemmatic sequence of cells will necessarily have a unique limiting point which is also the limiting point of the corresponding regular gemmatic sequence of traces.

If PQ and RS be two non-adjacent traces of an elementary chain c, the corresponding elementary cells of c will entirely lie outside each other with no point common and consequently the limiting points of any two regular gemmatic sequences of cells derived from them will be entirely distinct.

An *elementary arc* may now be defined as the aggregate of limiting points of all possible regular gemmatic sequences of elementary cells $e_1, e_2, \ldots, e_r, \ldots$ belonging respectively to a regular gemmatic system

of cellular chains, $c_1, c_2, \ldots, c_r \ldots$. More briefly an elementary arc may be defined as the limit of a regular gemmatic system of cellular chains.

8. The following properties of an elementary linear chain are evident.

(*i*) *No straight line other than one passing through two consecutive vertices can meet an elementary chain closed by its base AB at more than two points.*

(*ii*) *The successive traces $AP_1, P_1P_2 \ldots, P_{n-1}B$ of an elementary chain meet when produced negatively and positively the sides AT and TB respectively of its principal cell at two ordered rows of points $A, U_1, U_2, \ldots, U_{n-1}$ and $V_{n-1}, V_{n-2}, \ldots V_1, B$.*

(*iii*) *Every part of an elementary chain is an elementary chain.*

(*iv*) *If a point P travels continuously from A to B along the chain, the distance AP continuously increases and distance BP continuously diminishes.*

The corresponding properties of an elementary arc may be rigorously deduced:

(*i*) *No straight line can meet an elementary arc in more than two points.*

(*ii*) *There exists a tangent at each point P of an elementary arc which changes its direction continuously in the same sense as P travels from A to B along the arc.*

(*iii*) *Every part of an elementary arc is an elementary arc.*

(*iv*) *If a point P travels continuously from A to B along the arc, the distance AP continuously increases and the distance BP continuously diminishes.*

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On the expansion of the Weierstrassian and Jacobian Elliptic Functions in powers of the argument

BY

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The object of the present paper is to obtain the expansions of $\wp(u)$, $\mathrm{sn}(u)$, $\mathrm{cn}(u)$ and $\mathrm{dn}(u)$ in ascending series of u. It is believed that no previous writer has obtained the expansions of $\wp(u)$, $\mathrm{sn}(u)$, $\mathrm{cn}(u)$ and $\mathrm{dn}(u)$ beyond u^{18}, u^{9}, and u^{8} respectively. I have, in this paper obtained the expansion of $\wp(u)$, $\mathrm{sn}(u)$, $\mathrm{cn}(u)$ and $\mathrm{dn}(u)$ as far as u^{34}, u^{21} and u^{18} respectively.

My best thanks are due to Dr. Ganesh Prasad, D.Sc. who kindly suggested the problem to me.

Expansion of $\wp(u)$.

1. We know that $\wp(u)$ satisfies the differential equation

$$\left(\frac{dy}{du}\right)^2=4y^3-g_2y-g_3. \qquad \dots \quad (1)$$

Differentiating (1) we get the differential equation

$$\frac{d^2y}{du^2}=6y^2-\frac{g_2}{2} \qquad \dots \quad (2)$$

To solve this equation we assume

$$y=\frac{1}{u^2}+c_2u^2+c_4u^4+c_6u^6+\dots+c_{2n}u^{2n}+\dots$$

Substituting in (2) we get the well-known relation

$$c_{2n}=\frac{3}{(n-2)(2n+3)}\left\{c_2c_{2n-4}+c_4c_{2n-6}+c_6c_{2n-8}+\dots\right.$$

$$\left.+c_{2n-4}c_2\right\}. \qquad \dots \quad (3)$$

Substituting for y in (1) and equating the co-efficients of u^2 and u^4, we get

$$c_2=\frac{g_2}{2^2.5}, \qquad c_4=\frac{g_3}{2^2.7}.$$

Hence from (3), we get

$$c_6=\frac{g_2^2}{2^4\cdot3\cdot5^2}.$$

$$c_8=\frac{3g_2g_3}{2^4\cdot5\cdot7\cdot11}$$

$$c_{10}=\tfrac{1}{13}\left\{\frac{g_2^3}{2^5\cdot3\cdot5^3}+\frac{g_3^2}{2^4\cdot7^2}\right\}$$

$$c_{12}=\frac{g_2^2g_3}{2^5\cdot3\cdot5^2\cdot7\cdot11}$$

$$c_{14}=\tfrac{1}{17}\left\{\frac{g_2^4}{2^8\cdot3\cdot5^3\cdot13}+\frac{3g_2g_3^2}{2^4\cdot7^2\cdot11\cdot13}\right\}$$

$$c_{16}=\tfrac{1}{19}\left\{\frac{29g_2^3g_3}{2^8\cdot5^3\cdot7\cdot11\cdot13}+\frac{g_3^3}{2^6\cdot7^3\cdot13}\right\}$$

$$c_{18}=\tfrac{1}{21}\left\{\frac{7g_2^5}{2^9\cdot3\cdot5^5\cdot13\cdot17}+\frac{97g_2^2g_3^2}{2^8\cdot5\cdot7\cdot11^2\cdot13\cdot17}\right\}$$

$$c_{20}=\tfrac{1}{23}\left\{\frac{389\,g_2^4g_3}{2^9\cdot3\cdot5^4\cdot11\cdot13\cdot17\cdot19}+\frac{123\,g_2g_3^3}{2^7\cdot5\cdot7^3\cdot11\cdot13\cdot17\cdot19}\right\}$$

$$c_{22}=\frac{1}{25}\left\{\frac{g_2^6}{2^{11}\cdot3^3\cdot5^3\cdot13^2\cdot17}+\frac{25\,g_2^3g_3^2}{2^4\cdot7^2\cdot11^2\cdot13^2\cdot17\cdot19}+\frac{15g_3^4}{2^8\cdot7^4\cdot13^2\cdot19}\right\}$$

$$c_{24}=\frac{1}{27}\left\{\frac{729\,g_2^5\,g_3}{2^{11}\cdot5^4\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23}+\frac{1431\,g_2^2\,g_3^3}{2^8\cdot7^3\cdot11^2\cdot13\cdot17\cdot19\cdot23}\right\}$$

$$c_{26}=\frac{1}{29}\left\{\frac{g_2^7}{2^{12}\cdot3\cdot5^7\cdot13^2\cdot17}+\frac{13647\,g_2^4\,g_3^2}{2^{11}\cdot5^3\cdot7\cdot11^2\cdot13^2 17\cdot19\cdot23}+\frac{6471\,g_2g_3^4}{2^8\cdot5^2\cdot7^3\cdot11\cdot13^2\cdot17\cdot19\cdot23}\right\}$$

$$c_{28}=\frac{1}{31}\left\{\frac{104003\,g_2^6\,g_3}{2^{12}\cdot3^3\cdot5^6\cdot7\cdot11\cdot13^2\cdot17\cdot19\cdot23}+\frac{399701\,g_2^3\,g_3^3}{2^9\cdot3\cdot5^2\cdot7^3\cdot11^3\cdot13^2\cdot17\cdot19\cdot23}+\frac{g_3^5}{2^8\cdot5\cdot7^5\cdot13^2\cdot19}\right\}$$

$$c_{30}=\frac{1}{33}\left\{\frac{2453\,g_2^8}{2^{10}\cdot3^3\cdot5^8\cdot13^2\cdot17^2\cdot29}+\frac{1006029\,g_2^5\,g_3^2}{2^{11}\cdot5^5\cdot7^2\cdot11\cdot13^2\cdot17^2\cdot19\cdot23\cdot29}+\frac{8105017\,g_2^2\,g_3^4}{2^{10}\cdot5^3\cdot7^4\cdot11\cdot13^2\cdot17^2\cdot19\cdot23\cdot29}\right\}$$

$$c_{32}=\frac{1}{35}\left\{\frac{49871\,g_2^7\,g_3}{2^{16}\cdot3^2\cdot5^6\cdot13^2\cdot17\cdot19\cdot23\cdot29\cdot31}+\frac{240263\,g_2^4\,g_3^3}{2^{14}\cdot3\cdot5^2\cdot7\cdot11^2\cdot13^2\cdot17\cdot19\cdot23\cdot29\cdot31}+\frac{3693\,g_2g_3^5}{2^{12}\cdot5\cdot7^3\cdot13^2\cdot17\cdot19\cdot23\cdot29\cdot31}\right\}$$

$$c_{34}=\frac{1}{37}\left\{\frac{427\,g_2^{9}}{2^{17}\cdot3^{6}\cdot5^{6}\cdot13^{3}\cdot17^{3}\cdot29}\right.$$

$$+\frac{30458088737\,g_2^{6}\,g_3^{2}}{2^{16}\cdot3^{2}\cdot5^{5}\cdot7\cdot11^{2}\cdot13^{3}\cdot17^{2}\cdot19^{2}\cdot23\cdot29\cdot31}$$

$$\left.+\frac{122378650673\,g_2^{3}\,g_3^{4}}{2^{13}\cdot5^{3}\cdot7^{3}\cdot11^{3}\cdot13^{3}\cdot17^{2}\cdot19^{2}\cdot23\cdot29\cdot31}+\frac{43g_3^{6}}{2^{12}\cdot7^{5}\cdot13^{3}19^{2}\cdot31}\right\}$$

Expansion of sn(u).

2. Fricke [1] has shown that

$$\frac{d^{2v+1}}{du^{2v+1}}\,\mathrm{sn}=\mathrm{cn.\,dn}\left\{a_0^{(v)}+3a_1^{(v)}\mathrm{sn}^2+5a_2^{(v)}\mathrm{sn}^4+\ldots\right.$$

$$\left.+(2v+1)a_v^{(v)}\mathrm{sn}^{2v}\right\},$$

and

$$\frac{d^{2v+2}}{du^{2v+2}}\,\mathrm{sn}=\mathrm{sn}\left\{a_0^{(v+1)}+a_1^{(v+1)}\mathrm{sn}^2+\ldots+a_{v+1}^{(v+1)}\mathrm{sn}^{2v+2}\right\}.$$

The consecutive co-efficients are connected by the relation

$$a_\mu^{(v+1)}=(2\mu-1)2\mu\,k^2a_{\mu-1}^{(v)}-(2\mu+1)^2(1+k^2)a_\mu^{(v)}$$

$$+(2\mu+2)(2\mu+3)a_{\mu+1}^{(v)}\quad\ldots\quad(4)$$

where k is the modulus.

If we assume that

$$\mathrm{sn}(u)=u+a_1\frac{u^3}{\lfloor 3}+a_2\frac{u^5}{\lfloor 5}+a_3\frac{u^7}{\lfloor 7}+\ldots$$

then
$$a_v=\left(\frac{d^{2v+1}}{du^{2v+1}}\,\mathrm{sn}\,u\right)_{u=0}$$

[1] Die Elliptischen Funktionen Und Ihre Anwendungen, Vol. I. pp. 396-408.

Therefore
$$a_v = a_0^{(v)}.$$
From (4), we have

$$a_0^{(1)} = -(1+k^2). \qquad a_1^{(1)} = 2k^2.$$

$$a_0^{(2)} = (1+14k^2+k^4), \qquad a_1^{(2)} = -20k^2(1+k^2),$$

$$a_2^{(2)} = 24k^4.$$

$$a_0^{(3)} = -(1+135k^2+135k^4+k^6)$$

$$a_1^{(3)} = k^2(182+868k^2+182k^4)$$

$$a_2^{(3)} = -840k^4(1+k^2), \qquad a_3^{(3)} = 720k^6.$$

$$a_0^{(4)} = (1+1228k^2+5478k^4+1228k^6+k^8)$$

$$a_1^{(4)} = -k^2(1640+26520k^2+26520k^4+1640k^6)$$

$$a_2^{(4)} = k^4(23184+82656k^2+23184k^4)$$

$$a_3^{(4)} = -60480k^6(1+k^2).$$

$$a_4^{(4)} = 40320k^8.$$

$$a_0^{(5)} = -(1+11069k^2+165826k^4+165826k^6+11069k^8+k^{10})$$

$$a_1^{(5)} = k^2(14762+719576k^2+2141436k^4+719576k^6+14762k^8)$$

$$a_2^{(5)} = -k^4(599280+5504400k^2+5504400k^4+599280k^6)$$

$$a_3^{(5)} = k^6(3659040+11309760k^2+3659040k^4)$$

$$a_4^{(5)} = -6652800k^8(1+k^2)$$

$$a_5^{(5)} = 3628800k^{10}$$

$$a_0^{(6)} = (1+99642k^2+4494351k^4+13180268k^6+4494351k^8 +99642k^{10}+k^{12})$$

$$a_1^{(6)} = -k^2(132860+18616780k^2+136168760k^4+136168760k^6 +18616780k^8+132860k^{10})$$

$$a_2^{(6)} = k^4(15159144 + 314906592k^2 + 775927152k^4 + 314906592k^6 + 15159144k^8)$$

$$a_3^{(6)} = -k^6(197271360 + 1377604800k^2 + 1377604800k^4 + 197271360k^6)$$

$$a_4^{(6)} = k^8(743783040 + 2110268160k^2 + 743783040k^4)$$

$$a_0^{(7)} = -(1 + 896803k^2 + 116294673k^4 + 834687179k^6 + 834687179k^8 + 116294673k^{10} + 896803k^{12} + k^{14})$$

$$a_1^{(7)} = k^2(1195742 + 472128924k^2 + 7700190402k^4 + 17995941256k^6 + 7700190402k^8 + 472128924k^{10} + 1195742k^{12})$$

$$a_2^{(7)} = -k^4(380572920 + 16760441880k^2 + 86764270320k^4 + 86764270320k^6 + 16760441880k^8 + 380572920k^{10})$$

$$a_3^{(7)} = k^6(10121070960 + 140168508480k^2 + 310222392480k^4 + 140168508480k^6 + 10121070960k^8)$$

$$a_0^{(8)} = (1 + 8071256k^2 + 2949965020k^4 + 47152124264k^6 + 109645021894k^8 + 47152124264k^{10} + 2949965020k^{12} + 8071256k^{14} + k^{16})$$

$$a_1^{(8)} = -k^2(10761680 + 11873174000k^2 + 408992300880k^4 + 1968219965680k^6 + 1968219965680k^8 + 408992300880k^{10} + 11873174000k^{12} + 10761680k^{14})$$

$$a_2^{(8)}=k^4(9528671904+859275897408k^2+8567597445984k^4$$
$$+17583505295232k^6+8567597445984k^8$$
$$+859275897408k^{10}+9528671904k^{12})$$

$$a_0^{(9)}=-(1+72641337k^2+74197080276k^4$$
$$+2504055894564k^6+11966116940238k^8$$
$$+11966116940238k^{10}+2504055894564k^{12}$$
$$+74197080276k^{14}+72641337k^{16}+k^{18})$$

$$a_1^{(9)}=k^2(96855122+297545001712k^2$$
$$+20979207152120k^4+192841163567248k^6$$
$$+387317355330668k^8+192841163567248k^{10}$$
$$+20979207152120k^{12}+297545001712k^{14}$$
$$+96855122k^{16})$$

$$a_0^{(10)}=(1+653772070k^2+1859539731885k^4$$
$$+128453495887560k^6+1171517154238290k^8$$
$$+2347836365864484k^{10}+1171517154238290k^{12}$$
$$+128453495887560k^{14}+1859539731885k^{16}$$
$$+653772070k^{18}+k^{20}).$$

Expansion of cn(u).

3. Fricke has also shown that

$$\frac{d^{2v}\mathrm{cn}}{du^{2v}}=\mathrm{cn}\left\{b_0^{(v)}+b_1^{(v)}\mathrm{cn}^2+b_2^{(v)}\mathrm{cn}^4+\ldots+b_v^{(v)}\mathrm{cn}^{2v}\right\}$$

$$\frac{d^{2v+1}\mathrm{cn}}{du^{2v+1}}=-\mathrm{sn}\,\mathrm{dn}\left\{b_0^{(v)}+3b_1^{(v)}\mathrm{cn}^2+5b_2^{(v)}\mathrm{cn}^4+\ldots\right.$$
$$\left.+(2v+1)b_v^{(v)}\mathrm{cn}^{2v}\right\}.$$

The relation between the co-efficients is given by the equation

$$b_{\mu}^{v+1}=-(2\mu-1)2\mu\, k^2 b_{\mu-1}^{(v)}+(2\mu+1)^2(2k^2-1)b_{\mu}^{(v)}$$

$$+(2\mu+2)(2\mu+3)(1-k^2)\, b_{u+1}^{(v)} \qquad \ldots \quad (5)$$

Let us assume

$$\operatorname{cn}(u)=1+b_1\frac{u^2}{\lfloor 2}+b_2\frac{u^4}{\lfloor 4}+b_3\frac{u^6}{\lfloor 6}+\ldots$$

Then $$b_v=\left(\frac{d^{2v}\operatorname{cn}u}{du^{2v}}\right)_{u=0}.$$

Therefore

$$b_v=b_0^{(v)}+b_1^{(v)}+\ldots+b_v^{(v)}$$

From (5) we have

$$b_0^{(1)}=-1+2k^2,\qquad b_1^{(1)}=-2k^2$$

$$b_0^{(2)}=1-16k^2+16k^4,\qquad b_1^{(2)}=20k^2-40k^4,\qquad b_2^{(2)}=24k^4.$$

$$b_0^{(3)}=-1+138k^2-408k^4+272k^6.$$

$$b_1^{(3)}=-182k^2+1232k^4-1232k^6,$$

$$b_2^{(3)}=-840k^4+1680k^6,\qquad b_3^{(3)}=-720k^6.$$

$$b_0^{(4)}=1-1232k^2+9168k^4-15872k^6+7936k^8,$$

$$b_1^{(4)}=1640k^2-31440k^4+84480k^6-56320k^8.$$

$$b_2^{(4)}=23184k^4-129024k^6+129024k^8$$

$$b_3^{(4)}=60480k^6-120960k^8$$

$$b_4^{(4)}=40320k^8$$

$$b_0^{(5)}=-1+11074k^2-210112k^4+729728k^6$$
$$-884480k^8+353792k^{10}.$$

$$b_1^{(5)}=-14762k^2+778624k^4-4388736k^6$$
$$+7220224k^8-3610112k^{10}.$$

$$b_2^{(5)}=-599280k^4+7302240k^6-18311040k^8+12207360k^{10}.$$

$$b_3^{(5)}=-3659040k^6+18627840k^8-18627840k^{10}.$$

$$b_4^{(5)}=-6652800k^8+13305600k^{10}.$$

$$b_5^{(5)}=-3628800k^{10}.$$

$$b_0^{(6)}=1-99648k^2+4992576k^4-32154112k^6+71997696k^8$$
$$-67104768k^{10}+22368256k^{12}.$$

$$b_1^{(6)}=132860k^2-19281080k^4+211964480k^6-657704320k^8$$
$$+774592000k^{10}-309836800k^{12}.$$

$$b_2^{(6)}=15159144k^4-375543168k^6+1811601792k^8$$
$$-2872117248k^{10}+1436058624k^{12}.$$

$$b_3^{(6)}=197271360k^6-1969418880k^8+4724628480k^{10}$$
$$-3149752320k^{12}.$$

$$b_4^{(6)}=743783040k^8-3597834240k^{10}+3597834240k^{12}.$$

$$b_5^{(6)}=1037836800k^{10}-2075673600k^{12}.$$

$$b_6^{(6)}=479001600k^{12}.$$

$$b_0^{(7)} = -1+896810k^2-121675512k^4+1429612624k^6$$
$$-5354318720k^8+8804878080k^{10}-6663150592k^{12}$$
$$+1903757312k^{14}.$$

$$b_1^{(7)} = -1195742k^2+479303376k^4-10078771152k^6$$
$$+53541906944k^8-112628381952k^{10}$$
$$+103028914176k^{12}-34342971392k^{14}.$$

$$b_2^{(7)} = -380572920k^4+18663306480k^6$$
$$-157611767040k^8+451425461760k^{10}$$
$$-519526425600k^{12}+207810570240k^{14}.$$

$$b_3^{(7)} = -10121070960k^6+180652792320k^8$$
$$-791454343680k^{10}+1221603102720k^{12}$$
$$-610801551360k^{14}.$$

$$b_4^{(7)} = -71293622400k^8+636366931200k^{10}$$
$$-1481339059200k^{12}+987559372800k^{14}.$$

$$b_5^{(7)} = -192518726400k^{10}+900842342400k^{12}$$
$$-900842342400k^{14}.$$

$$b_6^{(7)} = -217945728000k^{12}+435891456000k^{14}$$

$$b_7^{(7)} = -87178291200k^{14}.$$

$$b_0^{(8)} = 1-8071264k^2+3006463840k^4-65021410816k^6$$
$$+389937612544k^8-1016535248896k^{10}+1318216683520k^{12}$$
$$-839461371904k^{14}+209865342976k^{16}.$$

$$b_1^{(8)} = 10761680k^2-11948505760k^4+480457340160k^6$$
$$-4191655738880k^8+14168862976000k^{10}-22391218606080k^{12}$$
$$+16723673415680k^{14}-4778192404480k^{16}.$$

$$b_2^{(8)}=9528671904k^4-916447928832k^6+13006907011584k^8$$
$$-60637227491328k^{10}+121459387060224k^{12}$$
$$-109368927977472k^{14}+36456309325824k^{16}.$$

$$b_3^{(8)}=507349664640k^6-15536891784960k^8$$
$$+112165149358080k^{10}-303438672875520k^{12}$$
$$+342992859955200k^{14}-137197143982080k^{16}.$$

$$b_4^{(8)}=6341563388160k^8-94388904529920k^{10}$$
$$+387671067463680k^{12}-586564325867520k^{14}$$
$$+293282162933760k^{16}.$$

$$b_5^{(8)}=29711191910400k^{10}-246864012595200k^{12}$$
$$+562324886323200k^{14}-374883257548800k^{16}.$$

$$b_6^{(8)}=62245299916800k^{12}-284549942476800k^{14}$$
$$+284549942476800k^{16}.$$

$$b_7^{(8)}=59281238016000k^{14}-118562476032000k^{16}.$$

$$b_8^{(8)}=20922789888000k^{16}.$$

$$b_0^{(9)}=-1+72641346k^2-74778211008k^4+3025469414016k^6$$
$$-28552658908416k^8+111959522763264k^{10}$$
$$-222711776673792k^{12}+238165246869504k^{14}$$
$$-130899983007744k^{16}+29088885112832k^{18}.$$

$$b_1^{(9)}=-96855122k^2+298319842688k^4-23064734107520k^6$$
$$+324970275402752k^8-1676632135367168k^{10}$$
$$+4100525862551552k^{12}-5172757729771520k^{14}$$
$$+3262213521932288k^{16}-815553380483072k^{18}.$$

$$b_2^{(9)} = -238345937760k^4 + 44839699800000k^6 - 1050618700696320k^8$$
$$+ 7580061634736640k^{10} - 23693732940595200k^{12}$$
$$+ 36226011554611200k^{14} - 26748518378373120k^{16}$$
$$+ 7642433822392320k^{18},$$

$$b_3^{(9)} = -25145993724480k^6 + 1295113966410240k^8$$
$$- 14661508613921280k^{10} + 62388114416271360k^{12}$$
$$- 120332370011258880k^{14} + 106965975363747840k^{16}$$
$$- 35655325121249280k^{18}.$$

$$b_4^{(9)} = -542078215660800k^8 + 12811131585907200k^{10}$$
$$- 83396879858073600k^{12} + 216317767886438400k^{14}$$
$$- 241079771971584000k^{16} + 96431908788633600k^{18}.$$

$$b_5^{(9)} = -4165794926092800k^{10} + 55265922161049600k^{12}$$
$$- 216772856178278400k^{14} + 323013868034457600k^{16}$$
$$- 161506934017228800k^{18}.$$

$$b_6^{(9)} = -14441333018112000k^{12} + 114162961296384000k^{14}$$
$$- 255840885780480000k^{16} + 170560590520320000k^{18}$$

$$b_7^{(9)} = -24666923138457600k^{14} + 110832202594713600k^{16}$$
$$- 110832202594713600k^{18}$$

$$b_8^{(9)} = -20274183401472000k^{16} + 40548366802944000k^{18}$$

$$b_9^{(9)} = -6402373705728000k^{18}$$

We therefore have

$$a_1 = -(1+k^2), \qquad a_2 = (1+14k^2+k^4)$$

$$a_3 = -(1+135k^2+135k^4+k^6)$$

$$a_4 = (1+1228k^2+5478k^4+1228k^6+k^8)$$

$a_5 = -(1+11069k^2+165826k^4+165826k^6+11069k^8+k^{10})$

$a_6 = (1+99642k^2+4494351k^4+13180268k^6+4494351k^8$
$+99642k^{10}+k^{12})$

$a_7 = -(1+896803k^2+116294673k^4+834687179k^6+834687179k^8$
$+116294673k^{10}+896803k^{12}+k^{14})$

$a_8 = (1+8071256k^2+2949965020k^4+47152124264k^6$
$+109645021894k^8+47152124264k^{10}+2949965020k^{12}$
$+8071256k^{14}+k^{16})$

$a_9 = -(1+72641337k^2+74197080276k^4+2504055894564k^6$
$+11966116940238k^8+11966116940238k^{10}+2504055894564k^{12}$
$+74197080276k^{14}+72641337k^{16}+k^{18})$

$a_{10} = (1+653772070k^2+1859539731885k^4+128453495887560k^6$
$+1171517154238290k^8+2347836365864484k^{10}$
$+1171517154238290k^{12}+128453495887560k^{14}$
$+1859539731885k^{16}+653772070k^{18}+k^{20})$

$b_1 = -1, \qquad b_2 = 1+4k^2, \qquad b_3 = -1-44k^2-16k^4$

$b_4 = 1+408k^2+912k^4+64k^6$

$b_5 = -1-3688k^2-30768k^4-15808k^6-256k^8$

$b_6 = 1+33212k^2+870640k^4+1538560k^6+259328k^8+1024k^{10}$

$b_7 = -1-298932k^2-22945056k^4-106923008k^6-65008896k^8$
$-4180992k^{10}-4096k^{12}$

$b_8 = 1+2690416k^2+586629984k^4+6337665152k^6+9860488448k^8$
$+2536974336k^{10}+67047424k^{12}+16384k^{14}$

$b_9 = -1-24213776k^2-14804306080k^4-345558617984k^6$
$-1165333452544k^8-782931974144k^{10}-95153582080k^{12}$
$-1073463296k^{14}-65536k^{16}.$

Expansion of dn(u)

4. We can now expand dn u

Let us assume that

$$\operatorname{dn} u = 1 + c_1 \frac{u^2}{\lfloor 2} + c_2 \frac{u^4}{\lfloor 4} + c_3 \frac{u_6}{\lfloor 6} \cdots\cdots$$

Then c_1, c_2, c_3 etc. can be calculated by means of the relation

$$k^{2\nu} b_\nu \left(\frac{1}{k^2}\right) = c_\nu \ (k^2)$$

Therefore

$$c_1 = -k^2, \qquad c_2 = k^4 + 4k^2, \qquad c_3 = -k^6 - 44k^4 - 16k^2$$

$$c_4 = k^8 + 408k^6 + 912k^4 + 64k^2$$

$$c_5 = -k^{10} - 3688k^8 - 30768k^6 - 15808k^4 - 256k^2$$

$$c_6 = k^{12} + 33212k^{10} + 870640k^8 + 1538560k^6 + 259328k^4 + 1024k^2$$

$$c_7 = -k^{14} - 298932k^{12} - 22945056k^{10} - 106923008k^8$$
$$- 65008896k^6 - 4180992k^4 - 4096k^2$$

$$c_8 = k^{16} + 2690416k^{14} + 586629984k^{12} + 6337665152k^{10}$$
$$+ 9860488448k^8 + 2536974336k^6 + 67047424k^4 + 16384k^2$$

$$c_9 = -k^{18} - 24213776k^{16} - 14804306080k^{14} - 345558617984k^{12}$$
$$- 1165333452544k^{10} - 782931974144k^8 - 95153582080k^6$$
$$- 1073463296k^4 - 65536k^2.$$

3

Motion of the earth as conceived by the ancient Indian Astronomers.

BY

Sukumar Ranjan Das

It is now an established fact that the earth does rotate and that the apparent diurnal motion of the heavens from east to west appears from the following considerations :—

(1) From simplicity ;
(2) From analogy ;
(3) From centrifugal force ;
(4) From the experiment of letting a body fall from the top of a high tower ;
(5) From Foucault's pendulum experiment ;
and (6) From gyroscopic experiment.

The first three arguments show that it is extremely possible that the earth rotates ; but the last three give experimental proofs of its rotation. It was Copernicus who was the first to maintain in Europe that the earth rotates and his argument in favour of earth's rotation was that this was a much simpler, and therefore a much probable explanation than that all the stars and other heavenly bodies should be linked together in such a complicated manner as to perform each its revolution round the celestial pole in the same time. However, the subsequent invention of telescope by Galileo in 1609 established this beyond doubt in Europe. By means of the telescope we can see that many of the planets, as well as the sun and moon, are spherical bodies rotating about axes, from which we conclude the earth also rotates. It is now known that the earth completes a rotation round its axis in a period of 24 hours which is called its diurnal motion and it completes a revolution round the sun in an elliptic path in a period of 365 days, 5 hours, 47 minutes, 48 seconds which is called its annual motion. However, a long time elapsed before this fact was conclusively proved and the arguments that were put forward in favour of or against earth's rotation both in Europe and in India are not only instructive but also interesting.

Of the ancient Hindu astronomers Āryabhaṭa was the first to maintain the diurnal rotation of the earth round its axis. Āryabhaṭa flourished in the fifth century A.D., (C. 475 A.D.). In the first sloka of his Geetikāpāda he said that in the four Yugas *i.e.*, in 43,20000 solar years, the earth rotates 15822,7500 times. He illustrates his theory of earth's rotation in his Āryabhatiya (499 A.D.) by stating : *

"As a man on a boat in a river sees the immovable trees on either bank move westward, in a similar way the fixed stars appear to move westward with equal velocity to an observer on the equator."

But this theory of Āryabhata was not recognised by later Indian Astronomers till after a very considerable time. The first celebrated astronomer to oppose this theory was a contemporary of his, Varāhamihira. In his *Pañcha-siddhāntikā* he writes in refutation :—

"Others maintain that the earth revolves as if it were placed in a revolving engine, not the sphere ; if that were the case, falcons and other (winged creatures) could not return from the ether to their nests.

(Chap. XIII, sloka 6.)

"And, to mention another argument, if the earth revolved in one day, flags and similar things would, owing to the quickness of the revolution, stream constantly towards the west. If the earth, on the other hand, moves slowly, how does it revolve once within twenty four hours ?" (Sloka 7)

Even Lalla, who is said to have been a disciple of Āryabhata, could not subscribe to the theory of his master and in refutation he adduced one more argument in addition to those of Varāhamihira. He asked, " If the earth is moving at a very rapid speed, why does not an arrow projected upwards fall on the western side of the place of projection ? Why do not the clouds appear to move only towards the west ? You cannot say that the earth is moving at a less speed, as it has to complete one revolution round its axis in 24 hours." Brahmagupta and many other astronomers of later date put forward the above arguments against Aryabhata's theory of earth's rotation. Lalla probably lived in the sixth century A.D. and was a contemporary of Varāhamihira. Brahmagupta came after Varāhamihira and was born in 598 A.D. He wrote his Brahma Sphūtā Siddhānta in 628 A.D. It is no doubt surprising that they did not consider for a moment that the aerial

* This does not clear whether Āryabhaṭa had in mind only the *geocentric* rotation of the Earth or its *heliocentric* revolution as well. But as he has stated the periodic times of the sun, Earth, moon and other planets in the same breath, *either* kinds of motion of the Earth might have been meant. Brahmagupta, also, while refuting this theory of Āryabhaṭa, refers to both kinds of motion. (This is mentioned in the next page.)

fluid might revolve with the same velocity as that of the earth. To refute this theory of earth's rotation Brahmagupta produced another argument, viz., "if the earth moves a minute in a prāna, then whence and in what route does it proceed? If it revolves, then why do not lofty objects fall?" (Brahmasphūta Siddhānta).

Bhattotpala, the celebrated commentator, who flourished in the 10th century A.D., could not accept the theory of Āryabhata. He also subscribed to the views of Varāhamihira and Brahmagupta. Even Bhāscara, the gem among ancient mathematicians, disapproved of the theory of Āryabhata. He did not deal with this question at a considerable length. Living so late as in the twelfth century A.D. he was engrossed with other more important problems of nature and did not think it worthwhile to take up much of his time in determination of the rotation of earth; for he knew that for all practical purposes the same results would be arrived at if rotation of the earth and fixity of the sun and stars or its contrary were accepted. Bhascara's disapproval of the theory of Aryabhata is known by his dealing with this question in the first chapter of Golādhyāya and mainly by the quotation of the views of Lalla against Aryabhata's theory.

The oppositionists in this question were so strong that Parameswara, a commentator of Aryabhata, had to twist the meaning of the couplet in Aryabhatiya already referred to. He said, "The earth is really fixed, but some say that the earth moves and the stars are fixed." The reason may be that Parameswara came just after Bhāscara and at that time no body could boldly maintain earth's rotation.

However, Aryabhata had one supporter in Prithūdaca Swami, the celebrated commentator of Brahmagupta who strongly supported Aryabhata's theory of earth's rotation. He affirmed "The starry sphere is stationary, and the earth, making a revolution, produces the daily rising and setting of the stars and planets."* Prithūdaca Swami further replied to the objection of Brahmagupta by saying, "Aryabhata's opinion appears, nevertheless, satisfactory, since planets cannot have two motions at once: and the objection that lofty things would fall is contradicted, for every day the under-part of the earth is also the upper, since, wherever the spectator stands on the earth's surface, even that spot is the uppermost point." This much is known

* भूपञ्जर: स्थिरो भूरेवावृत्यावृत्य प्रातिदैवसिकौ
उदयास्तमयौ सम्पादयति नक्षत्रग्रहाणाम् ।

This passage is generally mistaken for Aryabhata's own writing, but it is Prithudaca's saying.

of Prithūdaca Swami that he flourished before Śrīpati who quotes Prithūdaca in his works. Śrīpati is known to have written his Siddhānta Śekhara about 962 Saka or 1038 A.D. Hence Prithudaca might have propounded his views in the later half of the tenth century.

"Though Aryabhata was the first among ancient Hindu astronomers to give a clear idea about the diurnal rotation of the earth on its axis," says Colebrooke, "Yet there are instances to show the existence of this theory even seven centuries earlier." However, this theory took time to be recognised.

The theory that the earth moves daily round an axis, and that it has a motion round the sun as a kind of centre, which is completed in a year, is a doctrine so far removed from the evidence of our senses and so contrary to our daily observations, that before the proofs are understood, if it is received at all, it will be received as a mere opinion of men better able to judge of such matters, which may or may not be true.*

It is not therefore astonishing that even in the times of Copernicus, Kepler and Galileo, a theory which was so contrary and so entirely opposed to that which had been so universally received, should have been met with ridicule, and even with the persecution of its authors and their followers and that Galileo had to pay the penalty of his head when he boldly asserted the motion of the earth.

Here are some of the arguments put forward by some of the later Hindu Astronomers against Aryabhata's theory of the diurnal motion of the earth :—

(1) If the earth was in motion and made a complete revolution round its supposed axis, then pulled by that tremendous speed the houses, temples and other buildings on the surface of the earth would have, no doubt, crumbled down every minute and been shattered to pieces.

(2) On account of the constant revolution of the earth men, beasts and other animals could not have remained standing, not to speak of moving from one place to another.

(3) In consequence of the revolution of the earth, there would have been constant earth-quake, the waters would have swollen, and rivers would have ceased to flow and there would have been no tides at all.

(4) It is seen that a heavy body when falling from even the highest peak of a mountain, always falls at the foot of the mountain. But how could that have been possible if the earth was in motion?

* Brennand's Hindu Astronomy.

The circumference of the earth is 25 thousand miles and the earth takes 24 hours to complete a revolution on its supposed axis; hence the earth moves $\frac{25000}{24}$ or about 1000 miles per hour or $16\frac{2}{3}$ miles per minute. Now if a heavy body takes 30 seconds to touch the ground by that time the earth has moved eight miles. Therefore, how can the body fall at the foot of the mountain?

(See Brahma Sphūta Siddhānta)

(5) If we throw a stone from the west to hit a body on the east, then the stone must miss its aim as the body on the surface of the earth has moved considerably, if the earth is in motion.

(6) It is seen that rainfall continues at a placc for several hours. It would have been impossible if the earth was in motion. For, we know the earth moves about 16 miles per minute and every minute the place where rainfall has begun will be shifted from its original position by a considerable distance,

(7) If the earth was in motion, then the birds leaving their nests and flying in the sky would not have found their homes back whenever they liked to return. For, the trees on which their nests were situated would have moved considerably from their original position every minute. No doubt they could find their nests in the same place if they returned just after 24 hours. But that is not always the case. (Lalla)

(8) To illustrate further the last argument, it is said that ants swimming in water are carried by the current of the water; and therefore it is certain that birds must always fly in the direction of the motion of the earth, *i.e.*, they would be carried away by the much quicker motion of the earth from west to east. As the velocity of the ant is small compared with that of the current, the ant cannot go against the current; the velocity of the bird is many times small and even negligible in comparison with that of the earth, hence on no account the bird can overpower the motion of the earth and fly from east to west.

These arguments, though fallacious, are surely an outcome of a searching inquisitive brain and require much scientific knowledge for a reply. A satisfactory reply would, no doubt, presuppose a thorough knowledge of the law of relative velocity.

About the cause of rotation it is known from the numerous quotations of Bhattotpala that "Aryabhata accounted for the diurnal rotation of the earth on its axis by a wind or current of ærial fluid (प्रवह वायु), the extent of which, according to the orbit assigned to it by him, was little more than one hundred miles from the surface of the earth." This cause of rotation whether of the earth or of the starry heaven as the

case might be was accepted by Varāhamihira and others. In fact, about the cause of rotation this was the unanimous opinion of both the schools.

Now let us consider the views of astronomers anterior to Aryabhata. We have from the quotations by Bhattotpala that Pulisa siddhanta maintained that the earth was spherical, fixed at the centre of the rotating starry heaven and the rotation was due to the driving force of Prabaha or ærial current. The same views were held by the Vasistha siddhanta and they were quoted by Bhattotpala in his commentary of Bṛhat Saṃhita. As regards the views of the Jaina astronomers we desire to deal with them at a considerable length, as some of them were very peculiar.

It is interesting to note here that there appears in the Aitareya Brahmin a passage which is certainly not less than 2000 years before the birth of Copernicus in which it is stated that "the sun never sets nor rises. When people think to themselves the sun is setting, he only changes about after reaching the end of the day and makes night below and day to what is on the other side. Then when people think he rises in the morning, he only shifts himself about after reaching the end of the night and makes day below and night to what is on the other side.*

Dr. Haug was the first among oriental scholars who drew attention to this passage. He said " This passage is of considerble interest, containing the denial of the existence of sun-rise and sun-set. The author ascribes a daily course to the sun, but supposes it to remain always in its high position on the sky, making sun-rise and sun-set by means of its own contrarieties." Refering to this passage also Monier Williams says, "We may close the subject of Brahmanas by paying a tribute of respect to the acuteness of the Hindu mind, which seems to have made some shrewd astronomical guesses more than 2000 years before the birth of Copernicus." †

We have also in Bishnupurana a passage of almost similar significance which says, " the place where the sun is visible for a time has the sun-rise there and the sun sets on that place at that time where the sun is not visible. In truth, the sun never rises nor sets; when the sun is visible, he is said to have arisen, when the sun is not visible, he is said to have set." ‡

* *Vide* (Up. III. II-1-3) Prof. Radha Krishnan, *Indian Philosophy*, page 29.

† *Indian Wisdom*,

‡ *Bishnupuran*, Chap. VIII, part II, Slokas 14 and 15.

The Jaina astronomers who lived long before the time of Aryabhata, nay who were probably the oldest in the field of Astronomy, the authors of Joytish Vedanga being excepted, had a peculiar conception of the causes of day and night. They had no notion of the diurnal motion of the earth and consequent causes of the rising and setting of stars and planets. Till very recently no work of the Jaina Astronomers was available. Their trend of arguments could only be gathered from a few quotations by Varāhamihira, Brahmagupta, Bhāscarāchārya and others, in course of their refutation of the former's theories. There occurs a passage in the 13th chapter of Varāhamihira's Pancha-siddhāntika which refers to the Jaina Astronomers :—

"If, in agreement with the doctrine of the Arhat, there were two suns and moons rising by turns, how then is it that a mark made in the polar constellation by means of a line drawn from the sun revolves within one day?" Hence Varāhamihira concludes that there is one complete revolution of the sun within 24 hours.*

We have also in the Dūshana adhyāya of Brahmagupta's Sphūtā Siddhānta a passage refuting the theories of the Jainas :—

"There are fifty four nakshatras, two risings of the sun; this which has been taught by Jeina is untrue, since the revolution of the polar fish takes place within one day."†

The only work on Jaina Astronomy which is now available is Suryaprajnapti, in a longsized manuscript form with loose pages. It is written in Jaina prakrt and divided into twenty books called prābhṛtas, some of these again into chapters called prābhṛta prābhṛtas. The exact date of this work is not known. Thibaut says that this book must have been written before the Greeks came to India, as there is no trace of Greek influence in this work. Anyhow it must have been written at least three or four centuries before Christ. It is generally believed that Pythagoras was the first to maintain that the earth is a sphere. But long before Pythagoras propounded this view, the author of Suryaprajnapti spoke of the earth as an immense circular flat, *i.e.*, spherical. "The earth," says Suryaprajnapti, "is considered to be an immense circulur flat consisting of a number of concentric rings, called *dvipas*, separated from each other by ringshaped oceans. In the centre of the earth stands Mount Meru : around it runs the first dvipa-Jambudvipa. It is

* Pancha Siddhantika, Chap. XIII, verse 8.

† भानि चतु: पञ्चाशद् द्वौ द्वावर्कोदयौ जिनोक्तं यत् ।
ध्रुवमत्स्यावर्त्तो भवति यतोऽह्ना तदसदस्मात् ॥

surrounded by a circular ocean, the water of which is salt (लवण समुद्रं). The southern segment of the Jambudvipa is occupied by the Bhārata-varsha, and the northern segment by the Airâvata-varsha."

Now to come to the point we were discussing, the Jainas accounted for the alternation of day and night by imagining that the daily changes were caused by the passage of "two suns and two moons, and a double set of stars, and minor planets round a pyramidial mountain, at the foot of which is the habitable earth." Colebrooke says, "They (Jainas) conceive the setting and rising of stars and planets to be caused by the mountain Sumeru and suppose three times the period of a planet's appearance to be requisite for it to pass round Sumeru and return to the place where it emerged. Accordingly they allot two suns, as many moons and an equal number of each planet, star and constellation to Jambudvipa; and that these appear on alternate days south and north of Meru."*

Malayagiri, the commentator of Suryaprajnapti, has very lucidly explained the matter in classical Sanskrit. He says, "The earth is called Jambudvipa. the northern segment is Airābatabarsha and the southern segment is Bhāratabarsha. The two suns rising first shine on the northern and southern parts of Jambudvipa respectively and then again rise to shine upon the eastern and western parts respectively. Thus while the suns shine on the northern aud southern parts, the eastern and western parts are enveloped in darkness and there at that time night prevails." †

Sripati (960 Saka) in his Siddhānta-Sekhara, in the 11th century A. D. speaks of the Jaina Astronomers in the following sloka :—

"The Jainas say that the earth is not fixed, but descends perpetually in space, there are two suns, two moons, two sets of stars and planets and the Meru is of pyramidial shape." ‡

We have already seen that Brahmagupta refuted this absurdity. His refutation was copied by Bhāscaracharya, who added to it the refutation of another notion of the Jaina Astronomers regarding the falling of the earth in space, founded upon the idea that the earth being heavy and without support, must perpetually descend. Bhāscaracharya lived in the beginning of the twelfth century A. D.

* *Asiatic Researches*, Vol. IX, p. 321.

† *Suryaprajnapti*, *Prabhṛta*, II, p. 47.

‡ अधः पतन्त्याः स्थितिरस्ति गौर्व्या नभस्यनन्ते ऽत्र वदन्ति जैनाः ।
द्वौ द्वौ रवीन्दू द्विगुणं भसंस्थां चतुर्भुजस्तम्भनिभं च मेरुम् ॥

The replies given by him in his Siddhānta Siromani are as follows :— *

"The earth stands firm by its own power without other support in space." (Verse 2).

(Here we may mention a fact that in the Puranas the serpent, Ananta, is supposed to be the supporter of this earth. This must be an allegory and means no doubt that the earth stands without support in space, as the meaning of Ananta (अनन्त) is also "space.")

"If there be a material support to the earth, and another upholder of that, and again another of this, and so on, there is no limit. If, finally, self-support must be assumed, why not assume it at the very beginning? Why not rocognise it in this multiform earth?" (verse 4).

"As heat is in the sun and fire, coldness in the moon, fluidity in water, hardness in iron, so mobility is in air, and immobility in the earth, by nature," (verse 5).

"The earth possessing an attractive force (like loadstone for iron, says the commentator on Bhāscara), draws towards itself any heavy substance situated in the surrounded atmosphere and that substance appears as if it fell. But where can the earth fall, in ethereal space which is the same and alike on all sides?" (verse 6).

There is a passage in this connection in Chapter I of Golādhyāya which refers to the Buddhist astronomers who, however, on observation of the revolution of the stars, acknowledged that the earth had no support but maintained that the earth fell in ethereal space. No work of any Buddhist astronomer is available and it is generally believed that in consequence of Lord Buddha's preaching against the astronomical science, perusal of which by the Buddhist monks was strictly forbidden, possibly no astronomical work was ever written by any Buddhist of ancient times. However, Bhāscara referred to some remarks of the Buddhists in the following passage:—

"Whence dost thou, O Bauddha, get this idle notion, that because any heavy substance thrown into the air falls to the earth, therefore the earth itself descends?" (verse 9).

Bhascara in his Bāsanā (वासना) vāsya, further, illustrates this point. "For," he says, "if the earth were falling, an arrow shot into the air would not return to the earth when the projectile force was exhausted, since both would descend. Nor can it be said that it moves slower and is overtaken by the arrow, for heaviest bodies fall quickest and the earth is heaviest." (Bāsanā Vāsya on verse 9).

* *Siddhanta Siromani—Golādhyāya*, Chap. I.

Bhattotpala also had an argument against the falling oi earth in space. He said, "stone thrown upwards falls to the earth. If it be said that being very heavy earth falls slower than the lighter stone, then it must be asked in which direction it is falling. It cannot be said that earth is falling downward, For downward is a relative term and in fact the earth cannot have a downward part. Nor can it be said that the earth is falling sidewise. Hence it must be admitted that the earth is standing at the centre of the starry heaven without any support by dint of its own special power. (विशिष्टशक्तियुक्तौ)

The long discussion on this subject is not only interesting but also instructive. There were various changes of views, of course, not without the best intentions of further research. Of this Dr. Kern said in his preface to Bṛhat Samhitā: "And in no branch of Sanskrit literature have changes been made so freely as in astronomical works. Not from unworthy motives; on the contrary, the Hindu astronomers were the only class of learned men in their country who had an idea of science being progresive, not stationary or retrogressive."

These views show a deep insight and penetration into astronomical and physical phenomena and presuppose a spirit of enquiry in no way inferior to that shown even by Newton when he proceeded to suggest that if the earth rotates from west to east, a body, on being let fall from a considerable height above the earth's surface, shou'd fall to the east of the vertical line. The arguments advanced by them are not only interesting and instructive, but also set people thinking on many of the problems of nature that are often met with in our daily life.

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4

ON THOMAE'S CRITERION

(*An Extract of a letter to Prof. Ganesh Prasad*)

BY

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I had read Dr. Narayan's papers before writing my own, but became interested in the problem from another angle. The close analogy between series and integrals suggested that the notion of the limit of a function at a point might be generalised much in the same way that the sum (C 1) of a series is the generalised limit of its sum. Replacing h by x,

$$S_n = a_1 + a_2 + \ldots + a_n$$

by $f(x)$ and integrating instead of summing, it is then natural to call

$$\lim_{x \to +0} \frac{1}{x} \int_0^x f(x)\, dx$$

the generalised limit of the function $f(x)$ at $x=0+$. If we did not know the corresponding result for summable series, we might then expect this definition to define a limit for functions which though not continuous at $x=0$, do not oscillate very rapidly in its neighbourhood. *e.g.* sin (log x). However in the case of series it is known that the generalised (C 1) limit fails to exist not only for very divergent series but also for those which are "nearly but not quite convergent." The point of my paper was to show that the same state of affairs holds in the case of functions, and for the same reason. Sin (log x) is then a function too nearly continuous for the generalised limit to exist, just as R $\Sigma \dfrac{1}{n^{1+ia}}$ is too nearly convergent to be summable (C 1)

Other theorems on summable series have their analogues for functions. *e.g.* Hardy's necessary and sufficient condition that a series be summable (C 1), that

$$a_1+a_2+\dots+a_n+n\left[\frac{a_{n+1}}{n+1}+\frac{a_{n+2}}{n+2}+\dots\dots\right] \longrightarrow A$$

The theorem of Thomæ's you mention is analogous to the trivial theorem that

$$\frac{s_1+s_2+\dots+s_n}{n} \longrightarrow 0 \text{ if } \Sigma\frac{s_r}{r} \text{ converges.}$$

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5

ON CERTAIN MANY-VALUED SOLUTIONS OF THE EQUATIONS OF ELASTIC EQUILIBRIUM AND THEIR APPLICATION TO THE PROBLEM OF DISLOCATION IN BODIES WITH CIRCULAR BOUNDARIES

BY

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§ 1.

It was first pointed out by Wiengarten* that in a body occupying a multiply-connected space there is a physical possibility of the displacements being expressed by many-valued functions. He interpreted these displacements as being due to dislocations produced in the body by cutting it along a system of barriers which should be put in to make the region occupied by the body simply-connected, and then joining the opposite faces by removal or insertion of thin slices of matter of the same kind as that of the body. The new body so formed would then be in a state of initial stress. The stress and therefore the strain in the body must be single-valued and continuous, but the displacements would be discontinuous in crossing a barrier. The theory was further developed by Timpe,† Volterra and Cesaro.‡ It has been proved that the displacement of matter on one side of the barrier relative to that on the other side is one possible in a rigid

* *Roma. Acc. Linc. Rend.* (Ser. 5.), t 10 (1 Sem), 1901, p. 57.

† Probleme d. Spannungsverteilung in ebenen Systemen einfach gelost mit Hilfed. Airyschen Funktion, *Gottingen Diss.*, Leipzig, 1905.

‡ *Paris Ann. Ec. norm.* (Ser. 3) t. 24, 1907, pp 40-517.

body.* Moreover the position of the barriers in the body which has suffered dislocation and is consequently in a state of initial stress is immaterial and in fact there is nothing to show the seat of dislocation. Volterra † has applied the solutions of the equations of equilibrium of a two-dimensional elastic system in polar coordinates to the problem of dislocation in a hollow cylinder bounded by two concentric circles. In the present paper, use has been made of the expression for the stress function χ in bipolar coordinates to solve the problem of dislocation in a hollow cylinder whose crossection is bounded by two non-concentric circles, the cases discussed being those of (1) parallel fissures and (2) wedge-shaped fissures.

§ 2.

Let us consider the substitution

$$\alpha+i\beta=\log\frac{x+i(y+a)}{x+i(y-a)}$$

Here

$$x=\frac{a\sin\beta}{\cosh\alpha-\cos\beta},\qquad y=\frac{a\sinh\alpha}{\cosh\alpha-\cos\beta}$$

and

$$\frac{1}{h^2}=\left(\frac{\partial x}{\partial\alpha}\right)^2+\left(\frac{\partial y}{\partial\alpha}\right)^2$$

$$=\frac{a^2}{(\cosh\alpha-\cos\beta)^2}.$$

Therefore

$$h=\frac{\cosh\alpha-\cos\beta}{a}$$

The curves α=constant are a set of co-axial circles having the points $(0, -a)$ and $(0, a)$ as limiting points and β=constant, another

* Weingarten. *oc. cit.* Love, *Theory of Elasticity* (3rd edition), 156 A.

† *loc. cit.*

set of circles passing through the limiting points and cutting the first set orthogonally. We have $\beta=0$ on the y-axis except on the portion between the limiting points where $\beta=\pi$.

In the case of plane strain, the displacements u, v are given in terms of the stress function χ by the formulæ *

$$\left.\begin{aligned} 2\mu u &= \frac{\mu}{\lambda+\mu} \cdot h \frac{\partial \chi}{\partial \alpha} - h \frac{\partial Q}{\partial \beta} \\ 2\mu v &= \frac{\mu}{\lambda+\mu} \cdot h \frac{\partial \chi}{\partial \beta} + h \frac{\partial Q}{\partial \alpha} \end{aligned}\right\} \quad \ldots \quad (1)$$

where

$$hQ = \frac{\lambda+2\mu}{2(\lambda+\mu)} \iint \left\{ \frac{\partial^2 (h\chi)}{\partial \alpha^2} - \frac{\partial^2 (h\chi)}{\partial \beta^2} - h\chi \right\} d\alpha d\beta \quad \ldots \quad (2)$$

The stresses are given by †

$$\left.\begin{aligned} \widehat{a.\alpha\alpha} &= \left\{ (\cosh \alpha - \cos \beta) \frac{\partial^2}{\partial \beta^2} - \sinh \alpha \frac{\partial}{\partial \alpha} \right. \\ &\qquad \left. - \sin \beta \frac{\partial}{\partial \beta} + \cosh \alpha \right\} (h\chi) \\ \widehat{a.\alpha\beta} &= -(\cosh \alpha - \cos \beta) \frac{\partial^2 (h\chi)}{\partial \alpha \partial \beta} \\ \widehat{a.\beta\beta} &= \left\{ (\cosh \alpha - \cos \beta) \frac{\partial^2}{\partial \alpha^2} - \sinh \alpha \frac{\partial}{\partial \alpha} \right. \\ &\qquad \left. - \sin \beta \frac{\partial}{\partial \beta} + \cos \beta \right\} (h\chi) \end{aligned}\right\} \quad \ldots \quad (3)$$

* Jeffery. "Plane Stress and plane strain in bipolar coordinates" *Phil. Trans. Roy. Soc.* Ser. A. Vol 221, pp 265-293.

† Jeffery, *loc. cit.*

The terms in χ that give many-valued displacements are *

$$h\chi=(A\cosh\alpha+B\sinh\alpha+C\cos\beta+D\sin\beta)\,\alpha$$
$$+(A'\cosh\alpha+B'\sinh\alpha+C'\cos\beta+D'\sin\beta)\,\beta. \quad \ldots \quad (4)$$

and the corresponding value of Q is given by

$$hQ=\frac{\lambda+2\mu}{\lambda+\mu}(A\cosh\alpha+B\sinh\alpha+C\cos\beta+D\sin\beta)\,\beta$$
$$-\frac{\lambda+2\mu}{\lambda+\mu}(A'\cosh\alpha+B'\sinh\alpha+C'\cos\beta+D'\sin\beta)\alpha \quad \ldots(5)$$

Writing only the many-valued terms in the displacements, we have

$$\left.\begin{aligned}
2\mu u=&-\frac{\lambda+2\mu}{\lambda+\mu}\beta\Big[-C\sin\beta+D\cos\beta\\
&-\frac{\sin\beta}{\cosh\alpha-\cos\beta}(A\cosh\alpha+B\sinh\alpha+C\cos\beta+D\sin\beta)\Big]\\
&+\frac{\mu}{\lambda+\mu}\beta\Big[A'\sinh\alpha+B'\cosh\alpha\\
&-\frac{\sinh\alpha}{\cosh\alpha-\cos\beta}(A'\cosh\alpha+B'\sinh\alpha+C'\cos\beta+D'\sin\beta)\Big]\\
2\mu v=&\frac{\lambda+2\mu}{\lambda+\mu}\beta\Big[A\sinh\alpha+B\cosh\alpha\\
&-\frac{\sinh\alpha}{\cosh\alpha-\cos\beta}(A\cosh\alpha+B\sinh\alpha+C\cos\beta+D\sin\beta)\Big]
\end{aligned}\right\}\ldots(6)$$

$$+\frac{\mu}{\lambda+\mu}\beta\Big[-C'\sin\beta+D'\cos\beta$$
$$-\frac{\sin\beta}{\cosh\alpha-\cos\beta}(A'\cosh\alpha+B'\sinh\alpha+C'\cos\beta+D'\sin\beta)\Big]$$

* Jeffery, *loc. cit.*

The discontinuities of the displacements at any point of a barrier are given by

$$
\left.
\begin{aligned}
u_1-u_0 &= -\frac{\pi(\lambda+2\mu)}{\mu(\lambda+\mu)}\Big[\quad C\sin\beta+D\cos\beta \\
&\quad -\frac{\sin\beta}{\cosh\alpha-\cos\beta}\,(A\cosh\alpha+B\sinh\alpha+C\cos\beta+D\sin\beta)\Big] \\
&\quad +\frac{\pi}{\lambda+\mu}\Big[A'\sinh\alpha+B'\cosh\alpha \\
&\quad -\frac{\sinh\alpha}{\cosh\alpha-\cos\beta}(A'\cosh\alpha+B'\sinh\alpha+C'\cos\beta+D'\sin\beta)\Big] \\
v_1-v_0 &= \frac{\pi(\lambda+2\mu)}{\mu(\lambda+\mu)}\Big[A\sinh\alpha+B\cosh\alpha \\
&\quad -\frac{\sinh\alpha}{\cosh\alpha-\cos\beta}\,(A\cosh\alpha+B\sinh\alpha+C\cos\beta+D\sin\beta)\Big]
\end{aligned}
\right\} \quad \dots (7)
$$

$$
\begin{aligned}
&+\frac{\pi}{\lambda+\mu}\Big[-C'\sin\beta+D'\cos\beta \\
&-\frac{\sin\beta}{\cosh\alpha-\cos\beta}\,(A'\cosh\alpha+B'\sinh\alpha+C'\cos\beta+D'\sin\beta)\Big]
\end{aligned}
$$

Calculating Δ and ϖ from u_1-u_0 and v_1-v_0 we find that

$$
\left.
\begin{aligned}
\Delta &= \frac{2\pi}{\lambda+\mu}\quad\frac{A'+C'}{a}\,. \\
\text{and}\qquad 2\varpi &= \frac{2\pi(\lambda+2\mu)}{\lambda+\mu}\quad\frac{A+C}{a}\,.
\end{aligned}
\right\} \quad \dots (8)
$$

Thus if $A'+C'=0$, u_1-u_0 and v_1-v_0 represent a displacement possible in a rigid body. Hence these solutions can be applied to the problem of dislocation.

§ 3.

(a) *Parallel fissures.*

Let the cylinder be bounded by the circles $\alpha=\alpha_1$ and $\alpha=\alpha_2$.

(1) Let us assume *

$$h\chi=-A\alpha \sinh \alpha+B\alpha (\cosh \alpha-\cos \beta)$$
$$+(A_1 \cosh 2\alpha+B_1+C_1 \sinh 2\alpha) \cos \beta \qquad \ldots \quad (9)$$

The many-valued terms in the displacements are given by

$$\left.\begin{aligned} u&=-\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\cdot\frac{A\beta \sinh \alpha \sin \beta}{\cosh \alpha-\cos \beta} \\ v&=-\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\cdot\frac{A\beta(1-\cosh \alpha \cos \beta)}{\cosh \alpha-\cos \beta} \end{aligned}\right\} \qquad \ldots \quad (11)$$

When $\beta=0$, we have

$$u=0 \text{ and } v=0.$$

When $\beta=2\pi$, we have

$$u=0 \quad \text{and} \quad v=\frac{\lambda+2\mu}{\mu(\lambda+\mu)} \pi A.$$

Thus u is continuous on the barrier $\beta=0$, but v decreases suddenly by the amount

$$\frac{\lambda+2\mu}{\mu(\lambda+\mu)} \pi A$$

as we cross it. There is therefore a dislocation due to removal of a slice of thickness

$$\frac{\lambda+2\mu}{\mu(\lambda+\mu)} \pi A.$$

along $\beta=0$ *i.e.*, the y-axis or any other line parallel to the y-axis.

Calculating the stresses from (9), we have

$$\left.\begin{aligned} a.\widehat{\alpha\alpha}&=A \sinh^2 \alpha-B \sinh \alpha \cosh \alpha+A_1 \cosh 2\alpha+B_1+C_1 \sinh 2\alpha \\ &\qquad+\sinh \alpha \cos \beta[B-2(A_1 \sinh 2\alpha+C_1 \cosh 2\alpha)] \\ a.\widehat{\alpha\beta}&=-(\cosh \alpha-\cos \beta)\sin \beta[B-2(A_1 \sinh 2\alpha+C_1 \cosh 2\alpha)] \end{aligned}\right\} \quad (12)$$

If the boundaries are free from stress, we have

* Jeffery, *loc. cit.*

$\widehat{\alpha\alpha}=0$ and $\widehat{\alpha\beta}=0$ when $\alpha=\alpha_1$ and when $\alpha=\alpha_2$.

Therefore

$$\left.\begin{aligned}
B&=A.\frac{\sinh^2\alpha_1-\sinh^2\alpha_2}{\sinh^2\alpha_1+\sinh^2\alpha_2}\cdot\frac{\cosh(\alpha_1-\alpha_2)}{\sinh(\alpha_1-\alpha_2)}.\\
A_1&=-\tfrac{1}{2}A.\frac{\sinh^2\alpha_1-\sinh^2\alpha_2}{\sinh^2\alpha_1+\sinh^2\alpha_2}\cdot\frac{\sinh(\alpha_1+\alpha_2)}{\sinh(\alpha_1-\alpha_2)}.\\
C_1&=\tfrac{1}{2}A.\frac{\sinh^2\alpha_1-\sinh^2\alpha_2}{\sinh^2\alpha_1+\sinh^2\alpha_2}\cdot\frac{\cosh(\alpha_1+\alpha_2)}{\sinh(\alpha_1-\alpha_2)}.\\
B_1&=\tfrac{1}{2}A+\tfrac{1}{2}A\ \frac{\sinh^2\alpha_1\sinh2\alpha_2-\sinh^2\alpha_2\sinh2\alpha_1}{\sinh^2\alpha_1+\sinh^2\alpha_2}\\
&\qquad\times\frac{\cosh(\alpha_1-\alpha_2)}{\sinh(\alpha_1-\alpha_2)}.
\end{aligned}\right\} \quad \ldots \quad (12)$$

(2) For the second case let us consider the solution *

$$h\chi=A\alpha\sin\beta+(A_1\sinh 2\alpha+B_1\cosh 2\alpha)\sin\beta \quad \ldots \quad (13)$$

The many-valued terms in the displacements are given by

$$\left.\begin{aligned}
u&=\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\cdot\frac{A\beta(1-\cosh\alpha\cos\beta)}{\cosh\alpha-\cos\beta}\\
v&=-\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\cdot\frac{A\beta\sinh\alpha\sin\beta}{\cosh\alpha-\cos\beta}
\end{aligned}\right\} \quad \ldots \quad (14)$$

The discontinuities at a barrier are given by

$$\left.\begin{aligned}
u_1-u_0&=\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\pi A.\frac{1-\cosh\alpha\cos\beta}{\cosh\alpha-\cos\beta}=\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\pi Am\\
v_1-v_0&=-\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\pi A.\frac{\sinh\alpha\sin\beta}{\cosh\alpha-\cos\beta}=\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\pi Al
\end{aligned}\right\} \quad \ldots \quad (15)$$

* Jeffery, *loc. cit.*

where l and m are the direction cosines of the normal to α=constant.

The component of the relative displacement parallel to the x-axis

$$=l(u_1-u_0)-m(v_1-v_0)=0$$

while the component perpendicular to the x-axis

$$=m(u_1-u_0)+l(v_1-v_0)$$

$$=\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\pi A.$$

The displacements, therefore, represent a dislocation due to removal of a slice of thickness $\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\pi A$ along a barrier parallel to the x-axis.

The stresses at any point are given by

$$\left.\begin{aligned} a.\widehat{\alpha\alpha} &= -\sinh\alpha\ \sin\beta[A+2(A_1\cosh 2\alpha+B_1\sinh 2\alpha)] \\ a.\widehat{\alpha\beta} &= -(\cosh\alpha-\cos\beta)\cos\beta[A+2(A_1\cosh 2\alpha \\ &\qquad\qquad +B_1\sinh 2\alpha)] \end{aligned}\right\} \quad \dots \quad (16)$$

If the boundaries are free from stress, we have

$$\widehat{\alpha\alpha}=0 \quad \text{and} \quad \widehat{\alpha\beta}=0 \quad \text{when} \quad \alpha=\alpha_1 \quad \text{and when} \quad \alpha=\alpha_2.$$

Then

$$\left.\begin{aligned} A_1 &= -\tfrac{1}{2}A\,\frac{\cosh(\alpha_1+\alpha_2)}{\cosh(\alpha_1-\alpha_2)}. \\ B_1 &= \tfrac{1}{2}A\,\frac{\sinh(\alpha_1+\alpha_2)}{\cosh(\alpha_1-\alpha_2)}. \end{aligned}\right\} \quad \dots \quad (17)$$

(*b*) *Wedge-shaped fissures.*

Assume in this case*

$$h\chi=-A\alpha\cosh\alpha+B\alpha(\cosh\alpha-\cos\beta)$$
$$+(A_1\cosh 2\alpha+B_1+C_1\sinh 2\alpha)\cos\beta \quad \dots \quad (18)$$

* Jeffery, *loc. cit.*

The many-valued terms in the displacements are given by

$$\left.\begin{aligned} u &= -\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\,A\,\frac{\beta\cosh\alpha\sin\beta}{\cosh\alpha-\cos\beta}. \\ v &= \frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\,A\,\frac{\beta\sinh\alpha\cos\beta}{\cosh\alpha-\cos\beta}. \end{aligned}\right\} \quad \ldots \quad (19)$$

When $\beta=0$, we have

$$u=0 \quad \text{and} \quad v=0.$$

When $\beta=2\pi$, we have

$$u=0 \quad \text{and} \quad v=\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\,\pi A\,\frac{\sinh\alpha}{\cosh\alpha-1}.$$

The discontinuity of v on the barrier $\beta=0$ is therefore

$$\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\cdot\frac{\pi A\sinh\alpha}{\cosh\alpha-1}=v_0 \text{ (say).}$$

When $\beta=0$, we have

$$y=y_0=\frac{a\sinh\alpha}{\cosh\alpha-1}$$

Therefore

$$\frac{v_0}{y_0}=\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\;\frac{\pi A}{a}.$$

This dislocation can be interpreted as a wedge-shaped fissure bounded by the planes

$$x=\pm\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\;\frac{\pi A}{a}\,y. \quad \ldots \quad (20)$$

The stresses are given by

$$\left.\begin{aligned} a.\widehat{\alpha\alpha} &= -A(\alpha-\sinh\alpha\cosh\alpha)-B\sinh\alpha\cosh\alpha \\ &\quad +A_1\cosh 2\alpha+B_1+C_1\sinh 2\alpha \\ &\quad +\sinh\alpha\cos\beta[B-2(A_1\sinh 2\alpha+C_1\cosh 2\alpha)] \\ a.\widehat{\alpha\beta} &= -(\cosh\alpha-\cos\beta)\sin\beta \\ &\quad \times[B-2(A_1\sinh 2\alpha+C_1\cosh 2\alpha)] \end{aligned}\right\} \quad \ldots \quad (21)$$

If the boundaries are free from stress, we have

$\widehat{\alpha\alpha}=0$ and $\widehat{\alpha\beta}=0$ when $\alpha=\alpha_1$ and when $\alpha=\alpha_2$.

Therefore

$$\left.\begin{aligned} B &= 2k\cosh(\alpha_1-\alpha_2) \\ A_1 &= -k\sinh(\alpha_1+\alpha_2) \\ C_1 &= k\cosh(\alpha_1+\alpha_2) \\ B_1 &= \tfrac{1}{2}A\,\mathrm{cosech}(\alpha_1-\alpha_2)(\sinh^2\alpha_1+\sinh^2\alpha_2)^{-1} \\ &\quad \times[\sinh(\alpha_1+\alpha_2)\{\alpha_2\cosh 2\alpha_1-\alpha_1\cosh 2\alpha_2 \\ &\quad +\tfrac{1}{2}\sinh 2(\alpha_1-\alpha_2)\}-2\sinh\alpha_1\sinh\alpha_2 \\ &\quad \times(\alpha_2\sinh 2\alpha_1-\alpha_1\sinh 2\alpha_2)], \end{aligned}\right\} \quad \ldots \quad (22)$$

where

$$k=-\tfrac{1}{2}A[(\alpha_1-\alpha_2)\,\mathrm{cosech}(\alpha_1-\alpha_2)-\cosh(\alpha_1+\alpha_2)] \\ \times(\sinh^2\alpha_1+\sinh^2\alpha_2)^{-1}.$$

I take this opportunity to express my thanks to Dr. N. R. Sen for constant guidance and help in course of the work.

6

Was Āryabhaṭa indebted to the Greeks for his Alphabetic System of expressing numbers?

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Āryabhaṭa's rule giving his alphabetic system of expressing numbers is as follows:—

"वर्गाक्षराणि वर्गेऽवर्गेऽवर्गाक्षराणि कात् ङ्मौ यः।
खद्विनवके स्वरा नव वर्गेऽवर्गे नवान्त्यवर्गे वा॥"

It may be translated thus:

Barga consonants from क (onwards) [should be used] in *barga* places (*i.e.*, places corresponding to the *barga* or square units 1, 10^2, 10^4, &c.) and *abarga* consonants in *abarga* places (*i.e.*, places corresponding to the *abarga* or non-square units 10, 10^3, 10^5, &c.). य (*ya*) [stands for] 30 (lit., 5 and 25). Vowels [should be used] in eighteen places, nine [vowels] (with distinctly different sounds) in *barga* places as well as in the (corresponding) *abarga* places. [Those] nine [vowels] [should be used] in higher places in a similar manner.

In the above translation words within brackets [ ] have been introduced to complete sentences and words within brackets ( ) by way of explanation. The above translation differs from that given by the late Dr. Fleet (*J. R. A. S.*, 1911, p. 115). He writes: "The concluding words "*navāntyavargé va*" or 'in the square immediately following the nine' that is 'in the tenth square place' are enigmatical. They seem to indicate a ninteenth place (the number belonging to which, the British trillion, would be square of the Vrinda No. 10) and nothing after it" (*ibid*, p. 120). The alleged enigmatical nature of the words disappears when it is noticed that *navāntyavargé* is not a single compound word (समासबद्धपद) but has been formed by joining the

words *nava* and *antyavargé* according to the rules of *svarasandhi* (स्वरसन्धि) or conjunction of vowels. *Antya* means 'following' or 'which comes after' and *varga* means 'a group of the same class.' So the compound word *antyavargé* means 'in the following group of the same class.' As a group of eighteen places has already been spoken of in the verse, the word *antyavargé* can only mean 'in the following group of eighteen places.' In Sanskrit the word *vā* (वा) is used in the sense of 'or' as well as in the sense of 'and.' Here it cannot mean 'or'; it must have been used in the sense of 'and.' The word *vā* has also the force of similarity (सादृश्य) (*sādṛśya*). Hence the suggestion is that the number of places is unlimited and that nine vowels are to be used, as explained before, in each group of eighteen places. In Sanskrit rules brevity is secured by making them highly suggestive and not at all explicit. Explanations and implications are supplied by the teacher or the commentator. Fleet translates the second line of the above verse thus: "the nine vowels (are used) in the two nines of spaces square (and) not square, or in the square immediately following the nine." How can nine vowels be used in 'the square following the nine' by which expression Fleet means the ninteenth place? Only one vowel is to be used in each place. This is sufficient to prove that Fleet's translation is incorrect.

It has been suggested* that Āryabhaṭa's system of expressing numbers has been derived from the alphabetic notation of the Greeks. The reason for the above suggestion has been given† by the late Dr. Fleet in the following words: "Knowing the Greek source of the greater part of the astronomy etc., which we have in the Āryabhaṭīya and subsequent works, we naturally think of the possibility of a similar origin for this system of numeration." Does the question of borrowing at all arise in connection with the original matter which is included with admittedly borrowed matter in the work of a western savant? Is it not sometimes found that even distinguished European critics cannot view the writings of an ancient Indian scholar in the same spirit in which they consider the writings of western scholars? In giving his value of π Āryabhaṭa has used, not the Greek myriad or any of its corrupted forms, but the word (अयुत) *ayuta* which is equivalent to the Greek myriad and has been in use in India at least since the time of the Vedas‡ (*i.e.* from before the Greek civilisation

* Sudhakara Dvivedi, *Gaṇaka Tarangini* (1892). p. 5; Kaye, *J. A. S. B.*, (1907), p. 478; Fleet, *J. R. A. S.*, (1911), p. 125.

† *J. R. A. S.* (1911). p. 125.

‡ Macdonell and Keith, *Vedic Index*, Vol. I, pp. 342 and 343.

came into being). Yet it is asserted* that the way in which Āryabhaṭa has expressed his value of π points indubitably to a Greek source on the alleged ground that the Greeks alone of all peoples used the myriad as their unit of the second order. But has any European critic ever thought of the possibility of the Greek myriad having been borrowed from India and of Pythagoras's calling ten, hundred, etc., units of the second, third, etc., course after the Vedas which contain the earliest record of the use of a strictly decimal system of numeration, the different units being 1, 10, and higher powers of 10 ? † The renowned German mathematician Euler included both original and borrowed matter in his writings. When " he employed practically the same method of solution "‡ of the so-called Pellian equation as was given some centuries before by the Hindu mathematicians, did any one naturally think of the possibility of Euler's indebtedness to India for the solution? On the contrary, an attempt has been made § to belittle the importance of the equation in order to prove that the Hindus achieved nothing of importance in the field of mathematics. It is out of place here to multiply such examples of which there are many. We are all seeking truth and our only business is to give neither credit nor discredit but to sift evidence carefully and honestly and present the truth, whole truth, and nothing but the truth. If we are indebted to others, we need not feel ashamed to acknowledge our debt and express our gratitude. Hindu S'āstras look upon ingratitude with abhorrence. Let us, therefore, begin with an open mind. Let no consideration of race or clime deter us from pursuing the path of truth. Let us not form a conclusion beforehand and then try to find evidence in its support. We must not, therefore, begin by supposing that Āryabhaṭa was indebted to any foreign source for his alphabetic system of expressing numbers.

For the purposes of examination of evidence to trace the source of Hindu mathematics, Mr. Kaye postulates || three criteria for reference. The third criterion has been formulated thus : " Priority of statement of a proposition does not necessarily imply its discovery." Previous use alone of an alphabetic notation by the Greeks should not, therefore, lead us to trace Āryabhaṭa's alphabetic system to a Greek source. Even if all of Āryabhaṭa's writings excepting this rule were

* Heath, *History of Greek Mathematics*, Vol. I, p. 234.

† *Yajurveda*, Chapter XVII. *Mantra* 2.

‡ Kaye, *J. R. A. S.*, (1910), p. 752.

§ *Ibid*, p. 754.

|| *Ibid.* p. 750.

definitely known to have been based on foreign sources, we would not be justified, on that ground alone, in holding that this rule was also borrowed from a foreign source. This single rule might be Āryabhaṭa's own. If the second verse of the Gaṇitapāda (*i.e.* Chapter II) of the Āryabhaṭīya be read with the corresponding verses of Mahāvīra and Bhāskara, it will be seen that it contains an exposition of the modern place-value decimal notation.* No one has hitherto claimed that the modern place-value notation was known to the Greeks or to any other non-Hindu people before the sixth century.† It cannot, therefore, be said that Āryabhaṭa's rule explaining the modern notation was borrowed from the Greeks, " knowing the Greek source of the greater part of the astronomy etc., we have in the Āryabhaṭīya and subsequent works. If we cannot trace this rule to a Greek or foreign source, why should we begin by supposing the Greek origin of Āryabhaṭa's alphabetic system ? We should wait and see the result of applying the other two criteria.

The second criterion runs thus: " While mathematical systems of independent growth will naturally have many points of similarity, yet differences are certain to occur; it is, indeed, impossible for two systems to grow up independently in exactly the same manner."

Now, the only point of similarity between the Greek and Āryabhaṭian alphabetic systems is that the first nine letters of the alphabet denote the first nine numbers in each case. Āryabhaṭa's system differs from the Greek system in every other particular. The principal points of difference may be stated as follows :

* This fact has so far escaped the notice of distinguished Oriental scholars owing to their attention having been too much attracted by Āryabhaṭa's alphabetic system. Accordingly Mr. Kaye has been inconsistent in translating Āryabhaṭa's verse giving the modern notation. Fleet perceived this inconsistency and tried to be consistent. But in doing so he has mistranslated the principle *sthānāt sthānaṁ daśagunaṁ syāt.*

† Mr. Kaye writes that Iamblichus (4th century) " had perfectly clear ideas on the value of position " and gives the following example in his support :

" If the digits of any three be added together, and the digits of their sum be added together, and so on, the final sum is six." (*J. A. S. B.*, 1907, p. 493).

Mr. Kaye attributes this example to Jamblichus and cites as reference Gow's History of Greek Mathematics. But a comparison of Mr. Kaye's statement of the example with the statements of Gow (page 98) and Heath (History of Greek Mathematics, Vol. I, pp. 114 and 115) will show (*i*) that Mr. Kaye is not justified in putting the example in the above form which seems to support his conclusion and (*ii*) that other peoples who regarded numbers as being made up of a certain number of units, a certain number of tens, a certain nnmber of hundreds, etc., had equally " clear" (?) ideas on the value of position.

(*a*) In the Greek system the second group of nine letters denoted the first nine multiples of ten and the third group of nine letters the first nine multiples of hundred. To express multiples of higher powers of ten, strokes or dots were used. Each stroke or dot indicated multiplication by 1,000. Fleet says* that Āryabhaṭa's system is certainly not an adaptation of this system of the Greeks, but that Āryabhaṭa derived his inspiration from another Greek system of expressing large numbers in which myriads used to be expressed by means of *two* letter-numerals, *viz.*, (*i*) a symbol M for a myriad and (*ii*) the already adopted symbol for the number indicating the multiple. In Indian Kharosthi and later Brahmi notations hundreds used to be expressed in this way. But in Āryabhaṭa's system each multiple of a power of ten was denoted by a *single* consonant-numeral combined with a vowel-sign. Fleet is right in holding† that in Āryabhaṭa's scheme the vowels had no numerical values in themselves but marked the places to which the consonants, etc. were to be referred. If they had any numerical value, they could be used to express component parts of a number where no confusion was likely to arise. For example, ट्य could be used to denote 170. But vowels as such were never used by Āryabhaṭa; and vowel-signs cannot stand by themselves. So when Fleet considers‡ that, like the alleged§ Greek forms βM, γM, δM, खु, गु, घु, denote respectively 10000×2, 10000×3, 10000×4, he falls into a serious error. Whence does he get two numerical symbols? His explanation is far from satisfactory. He simply contradicts himself. For he writes: "But neither have the consonants, etc. nor have the vowels any numerical value in themselves; it is only by the combination of them into syllables that values are arrived at."‖ His transliteration¶ of खु, गु, घु into *kh*U, *g*U, *gh*U which bear resemblance to the alleged Greek forms βM, γM, δM, is probably responsible for this error. It is owing to the possibility of this kind of error resulting from transliteration or transcription that Mr. Kaye is not prepared to place any reliance on evidence in manuscripts.** Fleet is not,

* *J. R. A. S.* (1911), p. 125.

† *Ibid*, p. 118.

‡ *Ibid*, p. 126.

§ Gow says (Hist. of Greek Mathematics, p. 42, foot-note 2) that, if M followed β, γ, δ, etc, it was often omitted and a dot substituted. Heath says (Hist. of Greek Mathematics, Vol. I, pp. 39 & 40) that β, γ, δ, etc. were written either over or after M.

‖ *J. R. A. S.* (1911), p. 118.

¶ *Ibid*, p. 126.

** *J. A. S. B.* (1907),

therefore, justified in holding* that the idea behind the alleged Greek notations βM, γM, δM, etc. also underlies the whole of Āryabhaṭa's system. If it be assumed for the sake of argument that like βM, γM, etc., *kh*U, *g*U, etc. stood for 10000×2, 10000×3, etc., is it necessary to go to the Greeks for the source of Āryabhaṭa's inspiration? It could easily come from the already existing Indian way of expressing numbers in words, which has ever been on the decimal scale. द्वे अयुते, त्रीणि अयुतानि (two *ayutas*, three *ayutas*), etc. could easily suggest those notations. Unlike the Greek notations, Indian notations before Āryabhaṭa expressed numbers as they were spoken.

(*b*) As has been already stated, Āryabhaṭa's vowel-signs are not numerical symbols but indicate places which the consonant-numerals occupy. Hence Āryabhaṭa expresses numbers by means of consonant-numerals and as many place-indicating vowel-signs. But the Greek system exclusively employs letter numerals only. Place-indicating signs are conspicuous by absence in the Greek system.

(*c*) Unlike the Greek system and the old and modern Indian systems† of notation and the Indian way of speaking numbers, Āryabhaṭa's system (*i*) recognises component parts of a number, which are higher multiples than the ninth of even powers of ten, (*ii*) makes no provision for expressing the first two‡ multiples of the odd powers of ten *as such*, and (*iii*) admits of a number being expressed in more ways than one. For example, (*i*) मि (*mi*) stands for twenty-five hundreds; (*ii*) two thousands cannot be expressed *as such*; it should be regarded as twenty hundreds before it can be expressed by नि (*ni*); and (*iii*) forty-five can be expressed by मन, भय, वफ, यष, or रङ.

(*d*) The Greek alphabetic system was an arithmetical notation *i.e.* was used in performing arithmetical operations. But Āryabhaṭa's system was not so. As an arithmetical notation it has many and grave defects and is, therefore, useless. Hence it has not been given any place in the arithmetical portion of the Āryabhaṭīya. Arithmetical

* *J. R. A. S.* (1911), p. 126.

† These systems do not include the so-called 'word-symbol notation' which is not notation at all but which gives numbers expressed in the modern notation by stating the digits (one or two at a time) beginning with the units' place.

‡ According to Āryabhaṭa's scheme there are no letter-numerals for 1 and 2 to occupy *abarga* places. It will be seen from what follows that for the figures 3, 4, 5, etc. up to 10 there are two sets of letter-numerals—one set for *barga* places and the other for *abarga* places.

operations could be more easily and rapidly performed with the previous Indian notations.* The only merit of Āryabhaṭa's system is its conciseness and it has been devised chiefly to secure brevity of the rules composed in verse.

Thus, Mr. Kaye's second criterion, instead of proving the Greek origin of Āryabhaṭa's alphabetic system, points to its independent growth.

Mr. Kaye states his first criterion as follows: "The evolution of mathematical ideas cannot proceed *per saltum* but must proceed in an orderly manner." Let us, therefore, see if Āryabhaṭa's alphabetic system can be shewn to have had an orderly growth in India.

Āryabhaṭa had undoubtedly been a student of Sanskrit grammar and metrics. In Sanskrit grammar single letters are used for two different purposes, *viz.*, (*i*) as a suffix (प्रत्यय, *e.g.*, क, ट, etc.) and (*ii*) as a संज्ञा or name for something to which frequent reference has to be made (*e.g.*, ङु, घि, टि, etc.). In Piṅgala's manual of metrics single consonants (*e.g.*, भ, स, ज, etc.) have been used for the second purpose. To be included in metrical composition numbers, must, of necessity, be expressed by word-numerals or letter-numerals. The study of Sanskrit grammar and metrics seems to have led the mathematical genius of Āryabhaṭa to use letters of the Devanāgarī alphabet for the sake of brevity, as it afterwards led the well-known grammarian Bopadeva to use these letters in shortening the Sanskrit grammar. The vowels were not suitable for this purpose as they often disappear and merge into unrecognisable forms owing to conjunction (सन्धि) which is an essential feature of Sanskrit. Āryabhaṭa had, therefore, no other alternative than to use consonants to express numbers. The modern place-value notation was known to Āryabhaṭa who classified the places as *barga* and *abarga*. Most probably phonetic resemblance was responsible for the rule "*barga* consonants should be placed in *barga* places and *abarga* consonants in *abarga* places." To indicate the *barga* or *abarga* place occupied by a consonant nothing could be more convenient than a vowel-sign. Hence vowel-signs have been used as place-indicators. Āryabhaṭa names the first ten places (ekam, dasa, etc. up to vrindam) only. The first ten vowels were perhaps intended to indicate these places. Here some difficulty presented itself. Of the first two vowels अ, आ, the second is a long अ. The vowels of each of the remaining four pairs have similar sounds, the first vowel being short and the second long. In books on Sanskrit

* Vide Sir Richard Temple's article on the Burmese system of arithmetic in the *Indian Antiquary*, Vol. XX, pp. 53-69.

grammar the vowels constituting each of the five pairs are called equal (समान) vowels. Hence Āryabhaṭa seems to have overlooked the distinction between long and short vowels and made the rule that

अ (or आ) should indicate the places of the units				...	$10^0, 10^1$;
इ (or ई)	...	...	...	...	$10^2, 10^3$;
उ (or ऊ)	...	...	...	...	$10^4, 10^5$;
ऋ (or ॠ)	...	...	...	...	$10^6, 10^7$;
लृ (or ॡ)	...	...	...	...	$10^8, 10^9$:

Thus, each of the five vowels, अ, इ, उ, ऋ, लृ, long or short, was assigned to a *barga* place and the next higher *abarga* place. This could not result in confusion, as *barga* places were to be occupied by *barga* consonants only and *abarga* places by *abarga* consonants only. Then by the principle of anology—a principle which is responsible for two serious mistakes * made by Āryabhaṭa—he also assigned the remaining four vowels ए, ऐ, ओ, औ (which have distinctly different sounds), each to two consecutive places. Hence the rule "vowels should be placed in eighteen places, nine vowels in *barga* places and the same nine vowels in *abarga* places." As, by his rule, ञ and न denote the first and the second multiple of 10, he used य, र, &c. up to ह for the 3rd, 4th, &c...up to the 10th multiple of 10. It should be noted here that only when the consonants य, र, ल, व, &c. are each associated with the vowel अ, they denote 30, 40, 50, 60, &c. Otherwise they stand for the numbers 3, 4, 5, 6, &c.—up to 10. Thus, यि means य्, (or 3) put in the thousands' place (the *abarga* place assigned to इ). If य् stood for 30, यि would denote 30 tens or 300. Fleet is not right in holding that in Āryabhaṭa's scheme consonants have no numerical value in themselves. The metre shows that this reading is correct. If he were right, ङ् in the rule "ङ्मौ यः" could not stand for 5.†

It will thus be seen that Āryabhaṭa's alphabetic scheme had an orderly growth in India.

As a result of the application of the three tests laid down by Mr Kaye, we are forced to the conclusion that Āryabhaṭa is not indebted to a foreign source for his alphabetic system of expressing numbers.

* Āryabhaṭa's rules for the volumes of a tetrahedron and a sphere. (Vide *J. R. A. S.*, 1911, page 118).

† For this interpretation I am indebted to Dr. Bibhutibhusan Datta.

Review

LEBESGUESCHE INTEGRALE UND FOURIERSCHE REIHEN

Von L. Schlesinger Ord. Professor an der Universität Giessen und A. Plessner, Dr. Phil. III Und 229 S. Walter de Gruyter & Co., Berlin und Leipzig, 1926.

This book by Professor Schlesinger and Dr. Plessner on Lebesgue's Integral and Fourier's series is one of the many useful mathematical books which post-war Germany has produced. As is well-known, the notion of an integral for certain elementary functions was in a sense known to Fermat, Bhaskara, and even Archimedes who gave the surface and volume of a sphere in terms of its radius. The notion was generalized by Leibnitz and Cauchy to cover every continuous function and was further generalized by Riemann to apply to a certain class of functions having infinite numbers of points of discontinuity. The notion of Lebesgue is the most general possible notion compatible with use in Geometry and Physics. Lebesgue's notion is applicable even to certain totally discontinuous functions, *i.e.*, functions which are nowhere continuous.

Before proceeding with an analysis of the contents of the book, it is desirable to elucidate by means of simple examples the notions of Cauchy, Riemann and Lebesgue. It should be clear that every bounded function which is integrable according to Riemann is also integrable according to Lebesgue, although the converse is not true. For, take a function $f(x)$ which is defined for every value of x in the interval $(0, 1)$ and which equals 1 for the points of the aggregate M_1 of rational numbers and equals 2 for the points of the aggregate M_2 of irrational numbers. The points of discontinuity of $f(x)$ have a measure 1 and therefore $f(x)$ has no integral according to Riemann. But according to Lebesgue's notion,

$$\int_0^1 f(x)dx = \int_{M_1} f(x)dx + \int_{M_2} f(x)dx$$

$$= 1.\text{measure of } M_1 + 2.\text{ measure of } M_2.$$

Now the measure of M_1 is zero and that of M_2 is 1. Therefore

$$\int_0^1 f(x)dx = 2.$$

But an unbounded function may be integrable according to Cauchy and still not integrable according to Lebesgue. For, take $f(x)$ to represent

$$x^2 \sin \frac{1}{x^2}$$

for $x \neq 0$ and to equal 0 for $x=0$. Then the function $f'(x)$ is integrable in the interval $(0, b)$ according to Cauchy's notion,

$$\int_0^x f'(x)\, dx \text{ being } x^2 \sin \frac{1}{x^2},$$

but is not integrable according to Lebesgue's notion. In fact

$$f'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} \quad \text{for} \quad x \neq 0,$$

$$f'(0) = 0.$$

Now $2x \sin \frac{1}{x^2}$ is bounded and has a Lebesgue integral but $\frac{1}{x} \cos \frac{1}{x^2}$, which for convenience may be denoted by $g(x)$, has no integral. For, if it had so would $|\, g \,|$ have an integral. But, taking a_n to represent

$$\{(n+\tfrac{1}{2})\pi\}^{-\frac{1}{2}}$$

and $b > 0$, we have for $p > 0$,

$$\int_0^b |\, g \,|\, dx = \int_{a_{p-1}}^b |\, g \,|\, dx + \sum_{n=p}^{\infty} \int_{a_n}^{a_{n-1}} |\, g \,|\, dx$$

which is impossible ; for, the series on the right is divergent, since

$$\int_{a_n}^{a_{n-1}} |\, g \,|\, dx = \left| \int_{a_n}^{a_{n-1}} g dx \right| = \left| \int_{a_n}^{a_{n-1}} x^2 \cos \frac{1}{x^2} \cdot \frac{dx}{x^3} \right|$$

$$> \frac{1}{(n+\frac{1}{2})\pi} \left| \int_{a_n}^{a_{n-1}} \cos \frac{1}{x^2} \cdot \frac{dx}{x^3} \right| = \frac{2}{(2n+1)\pi}$$

and the series

$$\sum_{n=p'}^{\infty} \frac{2}{(2n+1)\pi}$$

is divergent.

The book, which is based on the lectures delivered by Professor Schlesinger at the Giessen University, contains six chapters with the headings: Fundamental notions of the theory of aggregates, the measures of point-aggregates, on functions of real variables, the Lebesgue's integral, functions of one and of two variables, Fourier's series.

Probably the most suggestive results are contained in the fifth chapter which has articles headed: integral, interval-function, point-function of a variable; interval-functions of bounded value-sums; properties of monotone point-functions; the derivatives of a continuous function, a covering theorem; the derivatives of a continuous interval-function of bounded value-sums, singular and totally continuous part; resolution of an interval-function and of its total sums; two examples; determination of an interval-function by means of its derivatives; the primitive function and the indefinite integral; the theorem of Fubini; functions of two variables, the double integral; the double integral as point-function, derivation.

The article on the primitive function and the indefinite integral contains a clear account of possible cases. In general these cases are four and may be enumerated as follows:

A measurable function of one variable has

(1) an indefinite integral as well as a primitive function,

(2) an indefinite integral but no primitive function,

(3) a primitive function but no indefinite integral,

or (4) neither an indefinite integral nor a primitive function.

Every measurable and bounded function comes under Case (1), the function

$$\frac{d}{dx}\left(x^2 \sin \frac{1}{x^2}\right)$$

is an example of Case (3), the function $(1/x)$ for any interval including 0 is an example of Case (4), and Case (2) has been illustrated by a rather complicated example which need not be reproduced here.

If a function f possesses a primitive function, then the Lebesgue's integral of f furnishes the primitive function provided that f, leaving aside an aggregate of measure zero, is bounded or bounded on one side. If, on the contrary, f is unbounded on both sides

on an aggregate of measure greater than zero, then the primitive function can be represented by the Lebesgue's integral only then when f satisfies the conditions of integrability.

The last chapter contains some applications of Lebesgue's integral to the theory of Fourier's series. The seven articles of this chapter are headed as follows: some integral theorems; the Eulerian formulae, the theorem of Riemann and Lebesgue; convergence of Fourier's series; the convergence criterion of Lebesgue; the summation methods of Lebesgue and Fej'er; functions which are integrable along with their squares; the theorems of Parseval and that of Riess and Fischer.

A very interesting form of the convergence criterion is given in the fourth of the seven articles mentioned above. If

$$\phi(t) \quad \text{denote} \quad \tfrac{1}{2}\{f(x+t)+f(x-t)\}$$

then the convergence of the Fourier's series corresponding to $f(x)$ is assured provided that in addition to the continuity of $f(x)$ the following condition is satisfied: for every quantity $\eta > 0$ two positive quantities δ and ϵ_1 can be so chosen that, for $\epsilon < \epsilon_1$,

$$\int_{\epsilon}^{\delta} \left| \frac{\phi(t)-\phi(t+\epsilon)}{t} \right| dt < \eta.$$

Another interesting application of Lebesgue's notion is the well-known theorem: The Fourier's series corresponding to an integrable function $f(x)$ is almost everywhere summable according to Cesaro to $f(x)$.

By bringing out his lectures in the form of a book, Professor Schlesinger has placed the lovers of Higher Mathematics under as great an obligation as they were under by his giving them the famous treatise on the theory of linear differential equations. It is to be hoped that English knowing mathematicians will soon have an English translation which will make the contents of the book accessible to Indian beginners in the study of the theory of functions of real variables.

GANESH PRASAD